Hierarchies of Lexicographic Beliefs*

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Abstract. Lexicographic type structures (Brandendenburger, Friedenberg, Kiesler, ECMA 2008) have become a standard tool for the epistemic analysis of strategic reasoning in finite static games. Yet, the implicit approach of type structures does not allow to fully understand which limitations are imposed by different structures and how their choice influences epistemic characterization results. Here we start from hierarchies of lexicographic beliefs and construct two different canonical structures, one for all hierarchies and one for those which can be represented by a mutually singular LPS over strategy-hierarchy pairs of the opponent. It turns out that the latter includes also hierarchies where beliefs are not mutually singular at any order. Thus, mutual singularity is an infinite-order notion rather than a finite-order one. Moreover, we analyze the terminality properties of the canonical type structures, i.e. how they relate to other, generic lexicographic type structures. Our canonical structures are proved to be fundamental for the epistemic analysis of iterated admissibility in other papers (Catonini 2012, Catonini and De Vito, 2016)

Keywords: lexicographic probability systems, hierarchies of beliefs, lexicographic type structures, universality, terminality.

1 Introduction

Brandendenburger, Friedenberg and Kiesler [15] (henceforth, BFK) defined lexicographic type structures as the analogue of the traditional ones for lexicographic probability systems [7] (henceforth, LPS). Strictly speaking, they call lexicographic type structures those structures where the belief map associates each type with a *mutually singular* LPS over the cartesian product of the opponent's strategies and types. An LPS is mutually singular if each measure in the list assigns probability 1 to an event to which all other measures assign probability 0. Intuitevely speaking, mutual singularity means that each measure represents a conjecture over the space of uncertainty conditional on the realization of an event that has probability zero according to the previous measures; indeed, Blume, Brandenburger and Dekel [7], who axiomatize LPS's with and without mutual singularity, refer to mutually singular LPS's as Lexicographic *Conditional* Probability Systems. On the other hand, Dekel, Friedenberg and Siniscalchi [21] prove that the results of BFK hold through if the requirement of mutual singularity for LPS's is relaxed

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Therefore, in this paper, we construct and analyze both mutually singular and non-mutually singular structures.

Our aim is to take an explicit approach and identify all the hierarchies of lexicographic beliefs we are interested in, so to provide a synthetic representation of them in a canonical type structure. First, we do this operation starting from all hierarchies of lexicographic beliefs (satisfying a minimal *coherence* requirement). Second, we restrict attention to those hierarchies which admit a mutually singular LPS representation over strategy-hierarchy pairs of the opponent. Both choices turn out to be appropriate for different reasons. In the first case, we are able to obtain a universal type structure that encompasses all other lexicographic structures (with or without mutual singularity) as a belief-closed subspace. In the second case, we obtain a mutually singular structure that encompasses all other stuctures where the mutual singularity of the LPS's is not trivially obtained through redundancies. It is noteworthy that the mutually singular, canonical type structure includes hierarchies whose beliefs are not mutually singular at any order. This shows that mutual singularity is really an infinite order notion rather than a finite order one. Indeed, the inclusion of these hierarchies is crucial to obtain universality of the canonical structure in the class of all lexicographic type structures without redundancies, i.e., where each two types induce different hierarchies.

2 Preliminaries and notation

We begin with some definitions and the basic notation that will be used throughout the paper.[1] A measurable space is a pair (X, Σ_X), where X is a set and Σ_X is a σ-field, the elements of which are called *events*. When it is clear from the context which σ-field on X we are considering, we suppress reference to Σ_X and simply write X to denote a measurable space. Further, if X and Y are measurable spaces, and the function $f : X \to Y$ is measurable, we denote by $\sigma(f)$ the σ-field on X generated by f, i.e., $E \in \sigma(f) \subseteq \Sigma_X$ if and only if there exists $F \in \Sigma_Y$ such that $E = f^{-1}(F)$. All the sets considered in this paper are assumed to be metrizable topological spaces, and they are endowed with the Borel σ-field. A *Polish* space is a topological space which is homeomorphic to a complete, separable metrizable space. A *Lusin* space is a topological space which is the continuous, injective image of a complete, separable metrizable space.[2] Clearly, a Polish space is also Lusin. Every metrizable Lusin space is measure-theoretic isomorphic to a Borel subset of some Polish space.

If $\{X_n\}_{n\in\mathbb{N}}$ is a countable collection of pairwise disjoint topological spaces, then the set $X = \cup_{n\in\mathbb{N}} X_n$ is endowed with the *direct sum topology*.[3] The set X is metrizable Lusin (resp. Polish) provided each X_n is metrizable Lusin (resp. Polish). For a given family of mappings $\{f_n\}_{n\in\mathbb{N}}$, where $f_n : X_n \to Y$, let $f : X \to Y$ be the function defined as

$$f(x) = f_n(x), \ x \in X_n.$$

Following the terminology in [25], the map $f : X \to Y$ is called the *combination* of the functions $\{f_n\}_{n\in\mathbb{N}}$, and is often denoted by $\cup_{n\in\mathbb{N}} f_n$.

We consider any product, finite or countable, of topological spaces as a topological space with the product topology. As such, a countable product of metrizable Lusin (resp. Polish)

[1] A more detailed presentation of the following concepts, as well as related mathematical results, can be found in [8], [25], [46], [49], [52]. In the remainder of the paper, we shall make use of the results mentioned in this section, sometimes without referring to them explicitly.

[2] If X is a Lusin topological space, and Σ_X is the corresponding Borel σ-field, then the measurable space (X, Σ_X) is Standard Borel ([20, Proposition 8.6.13]).

[3] The assumption that the spaces X_n are pairwise disjoint is without any loss of generality, since they can be replaced by a homeomorphic copy, if needed (see [25, p.75]).

spaces is also metrizable Lusin (resp. Polish). Furthermore, given topological spaces X and Y, we denote by Proj_X the canonical projection from $X \times Y$ onto X; in view of our assumption, the map Proj_X is continuous and open (i.e., the image of each open set in $X \times Y$ is an open set in X under the map Proj_X). Finally, for a measurable space X, we denote by Id_X the identity map on X, that is, $Id_X(x) = x$ for all $x \in X$.

3 Hierarchies of lexicographic beliefs and lexicographic type spaces

3.1 Lexicographic probability systems

Given a topological space X, we denote by $\mathcal{M}(X)$ the set of Borel probability measures on X. The set $\mathcal{M}(X)$ is endowed with the *weak**-topology. Then, if X is metrizable Lusin (resp. Polish), then $\mathcal{M}(X)$ is also metrizable Lusin (resp. Polish).

Given a topological space X, let $\mathcal{N}(X)$ (resp. $\mathcal{N}_n(X)$) be the set of all finite (resp. length-n) sequences of Borel probability measures on X, that is,

$$\begin{aligned} \mathcal{N}(X) &= \cup_{n\in\mathbb{N}} \mathcal{N}_n(X) \\ &= \cup_{n\in\mathbb{N}} (\mathcal{M}(X))^n . \end{aligned}$$

Definition 1 *Call each $\overline{\mu} = (\mu_1, ..., \mu_n) \in \mathcal{N}(X)$ a* ***lexicographic probability system*** *(LPS). Say $\overline{\mu}$ is a* ***mutually singular*** *LPS if there are Borel sets $\{E_l\}_{l\leq n}$ in X such that, for every $l \leq n$, $\mu_l(E_l) = 1$ and $\mu_l(E_m) = 0$ for $m \neq l$. Write $\mathcal{L}(X)$ (resp. $\mathcal{L}_n(X)$) for the set of mutually singular (resp. length-n) LPS's.*

Both topological spaces $\mathcal{N}(X)$ and $\mathcal{L}(X)$ are metrizable Lusin provided X is metrizable Lusin (Lemma 8, Appendix 5.1.2).[4] In particular, if X is Polish, so are $\mathcal{N}(X)$ and $\mathcal{L}(X)$.[5]

For every Borel probability measure μ on a topological space X, the support of μ, denoted by $\mathrm{Supp}\mu$, is the smallest closed subset of X such that $\mu(\mathrm{Supp}\mu) = 1$. The support of a LPS $\overline{\mu} = (\mu_1, ..., \mu_n) \in \mathcal{N}(X)$ is thus defined as $\mathrm{Supp}\overline{\mu} = \cup_{l\leq n}\mathrm{Supp}\mu_l$.

Definition 2 *A LPS $\overline{\mu} = (\mu_1, ..., \mu_n) \in \mathcal{N}(X)$ is of* ***full-support*** *if*

$$\bigcup_{l\leq n} \mathrm{Supp}\mu_l = X.$$

Write $\mathcal{N}^+(X)$ (resp. $\mathcal{L}^+(X)$) for the set of full-support LPS's (resp. full-support mutually singular LPS's).

[4] If X is equipped with a metric, then the topology of $\mathcal{N}(X)$ can be generated by the same specific metric used by BFK (cf. [15, p.321]).

[5] BFK show that, under the assuption that X is Polish, $\mathcal{L}(X)$ is Borel in $\mathcal{N}(X)$ ([15, Corollary C.1]). Lemma 8 in Appendix 5.1.2 shows that a stronger statement holds true: $\mathcal{L}(X)$ is a G_δ-subset (i.e. a countable intersection of open subsets) of $\mathcal{N}(X)$, hence a Polish subspace of $\mathcal{N}(X)$ if X is Polish. Such result is not entirely new: A special case of Lemma 8 can also be deduced from an older result due to Burgess and Mauldin ([16, Theorem 2]). See Appendix 5.1.2 for further details.

Suppose we are given topological spaces X and Y, and a Borel map $f : X \to Y$. The map $\widetilde{f} : \mathcal{M}(X) \to \mathcal{M}(Y)$, defined by

$$\widetilde{f}(\mu)(E) = \mu\left(f^{-1}(E)\right), \mu \in \mathcal{M}(X), E \in \Sigma_Y,$$

is called the image (or pushforward) measure map of f. For each $n \in \mathbb{N}$, the map $\widehat{f}_{(n)} : \mathcal{N}_n(X) \to \mathcal{N}_n(Y)$ is defined by

$$(\mu_1, ..., \mu_n) \mapsto \widehat{f}_{(n)}((\mu_1, ..., \mu_n)) = \left(\widetilde{f}(\mu_k)\right)_{k \leq n}.$$

Thus the map $\widehat{f} : \mathcal{N}(X) \to \mathcal{N}(Y)$ defined by

$$\widehat{f}(\overline{\mu}) = \widehat{f}_{(n)}(\overline{\mu}), \overline{\mu} \in \mathcal{N}_n(X),$$

is called the **image LPS map of** f. In other words, the map $\widehat{f}$ is the combination of the functions $\left(\widehat{f}_{(n)}\right)_{n \in \mathbb{N}}$, and it is Borel measurable (Lemma 4).

In particular, if X and Y are metrizable Lusin spaces, then the marginal measure of $\mu \in \mathcal{M}(X \times Y)$ on X is defined as $\mathrm{marg}_X \mu = \widetilde{\mathrm{Proj}}_X(\mu)$. Consequently, the marginal of $\overline{\mu} \in \mathcal{N}(X \times Y)$ on X is defined as $\overline{\mathrm{marg}}_X \overline{\mu} = \widehat{\mathrm{Proj}}_X(\overline{\mu})$, and, by Lemma 4.(2) in Appendix 5.1.1, $\widehat{\mathrm{Proj}}_X : \mathcal{N}(X \times Y) \to \mathcal{N}(X)$ is a continuous, surjective and open map.

3.2 Hierarchies of lexicographic beliefs

Fix a two-players set I;[6] given a player $i \in I$, we denote by $-i$ the other player in I. For each $i \in I$, let S_{-i} be a non-empty space—called *space of primitive uncertainty*—describing aspects of the strategic interaction that player i is uncertain about. Throughout this paper, S_{-i} will represent player $-i$'s strategy set: Player i does not know which strategy player $-i$ is going to choose. Yet, other interpretations are also possible; for instance, S_{-i} may include player $-i$'s set of payoff functions, among which the true one is not known to player i. We assume that for each $i \in I$, S_{-i} is a metrizable Lusin space.

Each player $i \in I$ is endowed with a lexicographic belief on S_{-i}; such prior is called first-order lexicographic belief. However, first-order beliefs do not exhaust all the uncertainty faced by each player: Player i realises that player $-i$ has at least one first-order belief on S_i as well, and this belief is unknown to her. Thus, player i's second-order beliefs are represented by a LPS over S_{-i} and the space of $-i$'s first-order beliefs. Continuing in this fashion, each player is completely characterized by an infinite hierarchy of lexicographic beliefs.

Formally, for each $i \in I$ define inductively the collection of spaces $\left\{X_i^k\right\}_{k=0}^{\infty}$ as

$$X_i^0 = S_{-i}, \tag{3.1}$$

$$X_i^{k+1} = X_i^k \times \mathcal{N}\left(X_{-i}^k\right); k \geq 0. \tag{3.2}$$

An element $h_i^{k+1} = \left(\overline{\mu}_i^1, \overline{\mu}_i^2, ..., \overline{\mu}_i^{k+1}\right)$ is a $(k+1)$-order belief hierarchy, where $\overline{\mu}_i^k = \left(\mu_i^{k,1}, ..., \mu_i^{k,n}\right) \in \mathcal{N}\left(X_i^{k-1}\right)$ denotes i's k-order LPS, with $\mu_i^{k,q} \in \mathcal{M}\left(X_i^{k-1}\right)$ being the q-level of the k-order LPS. It is easily seen that, according to our notation,

$$X_i^{k+1} = X_i^0 \times \prod_{l=0}^{k} \mathcal{N}\left(X_{-i}^l\right).$$

[6] The analysis can be trivially extended to more than two players.

The set of all possible, infinite hierarchies of LPS's for player i is $H_i^0 = \prod_{k=0}^{\infty} \mathcal{N}\left(X_i^k\right)$. The space H_i^0 is endowed with the product topology, thus, according to Lemma 8 in Appendix 5.1.2, H_i^0 is a metrizable Lusin space.

The notion of coherence for hierarchies of beliefs (defined below) says that beliefs at different orders cannot contradict each other. To state this formally, let $\mathrm{Proj}_{X_i^{k-1}} : X_i^k \to X_i^{k-1}$ denote the coordinate projection, for all $k \geq 1$. Recall that the marginal of $\overline{\mu}_i^{k+1} \in \mathcal{N}\left(X_i^k\right)$ over X_i^{k-1}, viz. $\overline{\mathrm{marg}}_{X_i^{k-1}} \overline{\mu}_i^{k+1}$, is defined as the image LPS of $\overline{\mu}_i^{k+1}$ under $\mathrm{Proj}_{X_i^{k-1}}$, namely $\widehat{\mathrm{Proj}}_{X_i^{k-1}}\left(\overline{\mu}_i^{k+1}\right)$. Since each map $\mathrm{Proj}_{X_i^{k-1}}$ is onto, continuous and open (by definition of product topology), it follows from Lemma 4.(2) in Appendix 5.1.1 that so is the induced map $\widehat{\mathrm{Proj}}_{X_i^{k-1}}$.

Definition 3 *A hierarchy of beliefs $h_i = \left(\overline{\mu}_i^1, \overline{\mu}_i^2, ...\right) \in H_i^0$ is **coherent** if and only if, for each $k \geq 1$,*

$$\overline{\mathrm{marg}}_{X_i^{k-1}} \overline{\mu}_i^{k+1} = \overline{\mu}_i^k.$$

This definition of coherence is a simple generalization of the notion of coherence as in [39] or [14]; the two notions coincide if each $\overline{\mu}_i^k$ is a standard probability measure (i.e. a length-1 LPS). Note that a hierarchy of beliefs satisfying this coherence requirement consists of an infinite sequence of LPS's of the same length.[7]

We now introduce the concepts of mutual singularity and full-support for hierarchies of LPS's.

Definition 4 *Say $h_i = \left(\overline{\mu}_i^1, \overline{\mu}_i^2, ...\right) \in H_i^0$ is **mutually singular (at order** k) if there exists $k \geq 1$ such that $\overline{\mu}_i^k$ is mutually singular.*

Definition 5 *Say $h_i = \left(\overline{\mu}_i^1, \overline{\mu}_i^2, ...\right) \in H_i^0$ is **of full-support at order** $k \geq 1$ if $\overline{\mu}_i^k$ is a full-support LPS. Say h_i is **of full-support** if, for all $k \geq 1$, $\overline{\mu}_i^k$ is of full-support.*

The relation between coherent belief hierarchies and the notions of mutual singularity and full-support is given in the following proposition, which exhibits an interesting "duality":

Proposition 1 *Fix a coherent hierarchy $h_i = \left(\overline{\mu}_i^1, \overline{\mu}_i^2, ...\right) \in H_i^0$.*

1. *If h_i is mutually singular at order $k \geq 1$, then h_i is mutually singular at order k', for all $k' \geq k$.*
2. *If h_i is of full-support at order $k \geq 1$, then h_i is of full-support at order k', for all $k' \leq k$.*

[7] As we shall see below, from any type in a lexicographic type structure we can derive a corresponding coherent hierarchy with the property of all orders of beliefs being of the same length.

Proof: Part 1 follows from Lemma 5.(1) in Appendix 5.1.1. Since each coordinate projection $\mathrm{Proj}_{X_i^{k-1}} : X_i^k \to X_i^{k-1}$ is a continuous surjection, Part 2 follows from Lemma 7.(1) Appendix 5.1.1. ■

For each player $i \in I$, the space of all coherent hierarchies of beliefs is denoted by H_i^1. For each $i \in I$, write $\widetilde{\Lambda}_i^0$ for the set of mutually singular hierarchies of LPS's, and write $\widetilde{\Lambda}_i^1 = \widetilde{\Lambda}_i^0 \cap H_i^1$ for the set of mutually singular and coherent hierarchies of LPS's. Mutually singular and coherent hierarchies where mutual singularity is satisfied at order 1 have already been the subject of research in epistemic game theory—see [30] and [53].

Lemma 1 *For each $i \in I$,*

(i) *H_i^1 is a closed subset of H_i^0.*

(ii) *The set $\widetilde{\Lambda}_i^1$ is a Borel subspace of H_i^1, and it is a Polish subspace of H_i^1 provided each S_i is Polish.*

Our primary focus will be on hierarchies of beliefs which satisfy coherence and mutual singularity. The following lemma plays the central mathematical role in the construction of the canonical hierarchic space in the next section.

Lemma 2 *Fix a countable collection of Lusin spaces $\{W_l\}_{l\geq 0}$, and, for each $k \geq 0$, let $Z_k = \Pi_{l=0}^{k} W_l$. Fix a sequence of LPS's $(\overline{\nu}^k)_{k\in\mathbb{N}}$ where, for each $k \geq 1$, $\overline{\nu}^k \in \mathcal{N}(Z_{k-1})$ and $\overline{\mathrm{marg}}_{Z_{k-1}} \overline{\nu}^{k+1} = \overline{\nu}^k$. Thus, there exists a unique LPS $\overline{\nu}$ on $Z = \Pi_{l=0}^{\infty} W_l$ such that*

$$\overline{\mathrm{marg}}_{Z_{k-1}} \overline{\nu} = \overline{\nu}^k,\ \forall k \geq 1.$$

Furthermore,

1. *If there is $k^* \geq 1$ such that $\overline{\nu}^{k^*}$ is mutually singular, then $\overline{\nu}$ is mutually singular.*
2. *$\overline{\nu}$ is of full-support if and only if, for each $k \geq 1$, $\overline{\nu}^k$ is of full-support.*

Lemma 2 is essentially a version of the Kolmogorov Extension Theorem for LPS's (cf. [14, Lemma 1]), and its proof is relegated in Appendix. It is noteworthy that the reverse implication of part 1 of Lemma 2 is *not* true. That is, the LPS $\overline{\nu} \in \mathcal{N}(Z)$ could be mutually singular, even though every LPS $\overline{\nu}^{k+1} \in \mathcal{N}(Z_k)$ does not satisfy an analogous requirement.[8] This fact will play a crucial role in the construction of a canonical hierarchic space consistent with mutual singularity, as we will see in the next section.

[8] Consider the case in which the lenght of each LPS is 2. Examples where the reverse implication of Lemma 2.1 does not hold can be found in statistical inference and in the convergence theory of set martingales (see [47, Chapter 9] for a modern treatment). This literature goes back, at least, to the pioneering contribution of Kakutani [32], the so-called Dicothomy Theorem for infinite product measures.

3.3 The canonical hierarchic space(s)

In this section, we construct the *canonical hierarchic space*, that is, the space of all hierarchies of lexicographic beliefs consistent with (common belief of) coherence (and mutual singularity, in a second step). To this end, we first show that a coherent hierarchy for a player is equivalent to a belief over the cartesian product of his own space of primitive uncertainty and opponents' hierarchies. So we start from the following result (cf., [14, Proposition 1]).

Proposition 2 *For each $i \in I$, there exists a homeomorphism $f_i : H_i^1 \to \mathcal{N}\left(S_{-i} \times H_{-i}^0\right)$ such that*

$$\overline{\operatorname{marg}}_{X_i^{k-1}} f_i\left(\left(\overline{\mu}_i^1, \overline{\mu}_i^2, ...,\right)\right) = \overline{\mu}_i^k, \quad \forall k \geq 1.$$

Proof: Note that, for each $i \in I$, the set $S_{-i} \times H_{-i}^0$ can be written as

$$S_{-i} \times H_{-i}^0 = X_i^{k-1} \times \prod_{l=k-1}^{\infty} \mathcal{N}\left(X_{-i}^l\right).$$

We denote by $\operatorname{Proj}_{X_i^{k-1}}$ the projection map from $S_{-i} \times H_{-i}^0$ onto X_i^{k-1}. For each $i \in I$, let $\Phi_i : \mathcal{N}\left(S_{-i} \times H_{-i}^0\right) \to H_i^1$ be the "diagonal" map[9] defined by

$$\begin{aligned} \overline{\mu}_i \quad \longmapsto \quad & \left(\Phi_i^k\left(\overline{\mu}_i\right)\right)_{k\geq 1} = \left(\widehat{\operatorname{Proj}}_{X_i^{k-1}}\left(\overline{\mu}_i\right)\right)_{k\geq 1} \\ = \quad & \left(\overline{\operatorname{marg}}_{X_i^{k-1}}\overline{\mu}_i\right)_{k\geq 1}. \end{aligned}$$

The existence of the map Φ_i follows from Lemma 2. To see this, in Lemma 2 set $W_0 = X_i^0$ and $W_l = \mathcal{N}\left(X_{-i}^{l-1}\right)$ for all $l \geq 1$. So $Z_k = \Pi_{l=0}^k W_l = X_i^k$ for each $k \geq 0$, and $Z = S_{-i} \times H_{-i}^0$. Since X_i^0 is Lusin, it follows from an iterated application of Lemma 8 that all the Z_k's and Z are Lusin spaces. Thus each hierarchy $h_i \in H_i^0$ defines a sequence of LPS's over Lusin spaces, and the conditions of Lemma 2 are satisfied.

Since $\operatorname{Proj}_{X_i^{k-1}}$ is a continuous, open surjection between Lusin spaces, it follows from Lemma 4.(2) that each Φ_i^k is a continuous, open surjection from $\mathcal{N}\left(S_{-i} \times H_{-i}^0\right)$ to $\mathcal{N}\left(X_i^{k-1}\right)$. Continuity of each Φ_i^k implies continuity of the map Φ_i (cf. [42, Theorem 19.6] or [25, p.79]). By Lemma 2 the map Φ_i is a bijection, so there exists some $k \geq 1$ for which $\widehat{\operatorname{Proj}}_{X_i^{k-1}}$ is injective—hence, in view of the above, a continuous open bijection onto its image. By the Diagonal Theorem ([25, Theorem 2.3.20]), it turns out that, for each $i \in I$, Φ_i is a continuous open bijection, i.e., a homeomorphism. To conclude the proof, set $f_i = \Phi_i^{-1}$. ■

The homeomorphism just described implies that a player i's coherent hierarchy of LPS's determines his LPS over player $-i$'s hierarchies of beliefs. However, even if player i's hierarchy $h_i \in H_i^1$ is coherent, $f_i\left(h_i\right)$ could deem possible an incoherent hierarchy of the other player, that is, player i may believe (in an appropriate sense defined below) it is possible that player $-i$'s hierarchy is not coherent. We consider the case in which there is *common full belief of coherence.*

Formally, we say that player i, endowed with a coherent hierarchy h_i, **fully believes** an event $E \subseteq S_{-i} \times H_{-i}^0$ if $f_i\left(h_i\right)(E) = \overrightarrow{1}$, where $\overrightarrow{1}$ denotes a finite sequences of 1s; that is

[9] Let $\{Z_n\}_{n\geq 1}$ be a sequence of sets, and let $f : X \to Y \subseteq \prod_{n=1}^{\infty} Z_n$ be the function defined by $f(x) = \left(f_1(x), f_2(x), ...\right)$, where $f_n : X \to Z_n$. The function f is called the *diagonal of the mappings* $\{f_n\}_{n\geq 1}$ in many standard texbooks in topology (e.g. [25]).

to say, every probability measure of the LPS $f_i(h_i) \in \mathcal{N}\left(S_{-i} \times H^0_{-i}\right)$ assigns probability 1 to E.[10] **Common full belief** of coherence is imposed by defining inductively, for each $i \in I$, the following sets:

$$\begin{aligned} H_i^{l+1} &= \left\{ h_i \in H_i^1 \left| f_i(h_i) \left(S_{-i} \times H^l_{-i} \right) = \overrightarrow{1} \right. \right\}, l \geq 1, \\ H_i &= \cap_{l \geq 1} H_i^l . \end{aligned}$$

The set $\Pi_{i \in I} H_i$ is naturally interpreted as the set of players' hierarchies such that each player fully believes that the other player's hierarchy is coherent, fully believes that the other player fully believes that his hierarchy is coherent, and so on. Proposition 3 below shows that common full belief of coherence closes the model, in the sense that each player's coherent hierarchy induces all possible beliefs over his own space of primitive uncertainty and opponents' hierarchies.

Proposition 3 *The restriction of f_i to H_i induces a homeomorphism $\overline{f}_i$ from H_i onto $\mathcal{N}(S_{-i} \times H_{-i})$.*

Proof: It is easily seen that

$$H_i = \left\{ h_i \in H_i^1 \left| f_i(h_i)(S_{-i} \times H_{-i}) = \overrightarrow{1} \right. \right\}.$$

Indeed, if $h_i \in H_i$, then by σ-additivity of LPS's it follows that

$$\begin{aligned} f_i(h_i)(S_{-i} \times H_{-i}) &= f_i(h_i) \left(S_{-i} \times \cap_{l \geq 1} H^l_{-i} \right) \\ &= \lim_{l \to \infty} f_i(h_i) \left(S_{-i} \times H^l_{-i} \right) \\ &= \overrightarrow{1}. \end{aligned}$$

On the other hand, suppose that $h_i \in H_i^1$, and $f_i(h_i)(S_{-i} \times H_{-i}) = \overrightarrow{1}$. Clearly, $h_i \in H_i = \cap_{l \geq 1} H_i^l$. The restriction of the homeomorphism f_i to H_i is hereditarily continuous, injective and open, so it remains to show that $f_i(H_i)$ is homeomorphic to $\mathcal{N}(S_{-i} \times H_{-i})$. But this follows from Lemma 9, so there exists a homeomorphism $\overline{f}_i$ from H_i onto $\mathcal{N}(S_{-i} \times H_{-i})$. ■

Herafter, we shall refer to the set $H = \Pi_{i \in I} H_i$ as the *canonical hierarchic space*. It should be noted that if a hierarchy $h_i \in H_i$ is mutually singular (Definition 4), then $\overline{f}_i(h_i)$ is a mutually singular LPS by Lemma 2, formally $\overline{f}_i(h_i) \in \mathcal{L}(S_{-i} \times H_{-i})$. As already remarked, the reverse implication is not true. Using the canonical homeomorphism of Proposition 3, let

$$\Lambda_i^1 = \left\{ h_i \in H_i \left| \overline{f}_i(h_i) \in \mathcal{L}(S_{-i} \times H_{-i}) \right. \right\}.$$

That is, Λ_i^1 is the set of all hierarchies consistent with common full belief of coherence that can be summarized by a mutually singular belief over $S_{-i} \times H_{-i}$. Hereafter, we shall refer to the set Λ_i^1 as the *set of hierarchies with a mutually singular representation.* In view of the above, Λ_i^1 *properly* includes the set $\widetilde{\Lambda}_i^1 \cap H_i$, i.e., the set of mutually singular hierarchies consistent with common full belief of coherence.

Clearly, if a hierarchy $h_i \in \Lambda_i^1$ is consistent with common full belief of coherence, then the induced LPS $\overline{f}_i(h_i)$ is mutually singular, but player i does not necessarily fully believe that his opponents hierarchies are mutually singular as well. We thus consider the case in which there is

[10] This notion of full belief for LPS has been given an axiomatic, preference-based treatment by BFK ([15, Proposition A.1]).

common full belief of the event "coherence and mutual singularity" among the players. We do this by first defining, for each $i \in I$, the map $g_i : \Lambda_i^1 \to \mathcal{L}(S_{-i} \times H_{-i})$ as $g_i = \left(\overline{f}_i\right)^{-1}$. Then, we define inductively, for each $i \in I$, the following sets:

$$\begin{aligned} \Lambda_i^{l+1} &= \left\{ h_i \in \Lambda_i^1 \left| g_i(h_i)\left(S_{-i} \times \Lambda_{-i}^l\right) = \overrightarrow{1} \right. \right\}, \, l \geq 1, \\ \Lambda_i &= \cap_{l \geq 1} \Lambda_i^l. \end{aligned}$$

The set $\Lambda = \Pi_{i \in I} \Lambda_i$ is referred to as the *canonical hierarchic space consistent with mutual singularity.* The following Proposition shows that a homeomorphism result, analogous to the one provided by Proposition 3, also holds for each space of hierarchies Λ_i.

Proposition 4 *The restriction of $\overline{f}_i$ to Λ_i induces a homeomorphism $\overline{g}_i : \Lambda_i \to \mathcal{L}(S_{-i} \times \Lambda_{-i})$.*

Proof: Using the same arguments as those in the proof of Proposition 3, it is immediate to check that

$$\Lambda_i = \left\{ h_i \in \Lambda_i^1 \left| g_i(h_i)(S_{-i} \times \Lambda_{-i}) = \overrightarrow{1} \right. \right\}.$$

The remainder of the proof is virtually identical to that of Proposition 3. ■

We conclude this section with a few remarks concerning the topological structure of the canonical hierarchic spaces H and Λ. Tipically, the literature on hierarchies of beliefs (e.g., [4], [14]) begins with an underlying space of uncertainty that is a Polish space. It then imposes the weak*-topology on the sets of beliefs which yields, by construction, a corresponding Polish space of hierarchies of beliefs. In the present context of lexicographic beliefs, if each space S_i is Polish, so are H and Λ—this is easily seen by using Lemma 1 in the base step and then proceeding by induction on the sets H_i^l and Λ_i^l. But a similar conclusion holds if each space S_i is simply metrizable Lusin; that is, the Lusin property of the topologies on both H and Λ is inherited from the topology on each space of primitive uncertainty.[11]

We mention a further topological property of the canonical hierarchic spaces under consideration: *Both H and Λ are not compact*, even if the underlying spaces of primitive uncertainty are compact (e.g., finite). To see this, note that $\mathcal{M}(X)$ is compact if X is compact, and this in turn implies that the spaces $\mathcal{N}_n(X)$ and $\mathcal{L}_n(X)$ are also compact for some *finite* $n \in \mathbb{N}$. But the same conclusion does not hold for the spaces $\mathcal{N}(X)$ and $\mathcal{L}(X)$.[12] By contrast, the canonical hierarchic spaces of both standard beliefs and conditional beliefs turn out the be compact metrizable if each space S_i is compact metrizable.

Finally, we point out that our topological assumptions imposed on the space of LPS's are "natural" in the sense that they do not alter the conceptually appropriate measure-theoretic structure on the space of belief hierarchies. To illustrate, fix an event $E \subseteq X$ and a number $p \in \mathbb{Q} \cap [0,1]$. Say that player i p-**believes** E for length-n LPS $\left(\mu_i^1, ..., \mu_i^n\right)$ if $\mu_i^l(E) \geq p$, for all $l \leq n$ (cf., [29] and [41]; if $p = 1$, this corresponds to the notion of full belief introduced before). The statement "player $-i$ p-believes the event E for some *finite* length LPS $\overline{\mu}_{-i}$" can be expressed by the set $b_n^p(E) = \left\{ \left(\mu_{-i}^1, ..., \mu_{-i}^n\right) \in \mathcal{N}(X) \left| \mu_i^l(E) \geq p, \forall l \leq n \right. \right\}$. To formalize higher

[11] It should also be noted that both the space of standard belief hierarchies in [14] and the space of conditional belief hiearchies in [4] are metrizable Lusin provided all sets of primitive uncertainty are assumed to be metrizable Lusin. The Kolmogorov Extension Theorem, which plays a central role in the construction of the canonical space, is indeed applicable even for the case in which every factor space is Lusin (or Souslin) - see Theorem 5 and subsequent discussion in the Appendix for details.

[12] This is an instance of a well-known mathematical fact (see [25, Theorem 2.2.3]): If $\{X_\theta\}_{\theta \in \Theta}$ is a family of non-empty compact spaces, then the direct sum $\cup_{\theta \in \Theta} X_\theta$ is compact if and only if the right-directed set Θ is finite.

order statements such as "player i p-believes that 'player $-i$ p-believes E'" we need to require that the set $b_n^p(E)$ be an event in $\mathcal{N}(X)$. Lemma 12 in Appendix 5.1.2 shows that, under our topological assumptions, this is indeed the case: The Borel σ-field on the space $\mathcal{N}(X)$ coincides with $\mathcal{A}_{\mathcal{N}(X)}$, which is exactly the σ-field generated by sets of the form

$$\{(\mu_1, ..., \mu_n) \in \mathcal{N}(X) \,|\mu_l(E) \geq p_l, \forall l \leq n\},$$

where $p_l \in \mathbb{Q} \cap [0,1]$ for all $l \leq n$, and E is an event in X.

3.4 Lexicographic type structures

The following definition is a natural generalization of the standard definition of epistemic type structure with beliefs represented by probability measures, i.e., length-1 LPS (cf. [29]).

Definition 6 *An $(S_i)_{i\in I}$-**based lexicographic type structure** is a structure $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i\in I}$, where*

1. *for each $i \in I$, T_i is a metrizable Lusin space;*
2. *for each $i \in I$, the function $\beta_i : T_i \to \mathcal{N}(S_{-i} \times T_{-i})$ is measurable.*

*We call each space T_i **type space** and we call each β_i **belief map**.[13] Members of type spaces, viz. $t_i \in T_i$, are called **types**. Say $t_i \in T_i$ is a **mutually singular** type if $\beta_i(t_i) \in \mathcal{L}(S_{-i} \times T_{-i})$. Say $t_i \in T_i$ is a **full-support** type if $\beta_i(t_i) \in \mathcal{N}^+(S_{-i} \times T_{-i})$. Each element $(s_i, t_i)_{i\in I} \in S \times T$ is called **state (of the world)**.*

A lexicographic type structure—or type structure, for short—formalizes Harsanyi's implicit approach to model hierarchies of beliefs. But clearly the canonical hierarchic space $H = \Pi_{i\in I} H_i$ constructed in the previous section gives rise to an $(S_i)_{i\in I}$-based lexicographic type structure $\mathcal{T}_u = \langle S_i, T_i, \beta_i \rangle_{i\in I}$, by setting $T_i = H_i$ and $\beta_i = \overline{f}_i$ for each $i \in I$. Hereafter, we shall refer to $\mathcal{T}_u = \langle S_i, H_i, \overline{f}_i \rangle_{i\in I}$ as the *canonical lexicographic type structure*.

The formalism of lexicographic type strucures was first introduced by BFK ([15, Section 7]) under the additional requirement that each belief is represented by a mutually singular LPS. The following definition translates their notion of type structure into our setting.

Definition 7 *Call a lexicographic type structure $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i\in I}$ **mutually singular** if, for each $i \in I$, every $t_i \in T_i$ is a mutually singular type. (I.e., the range of each belief map β_i is contained in $\mathcal{L}(S_{-i} \times T_{-i})$.)*

It is easily seen that also the canonical hierarchic space Λ gives rise to a type structure $\mathcal{T}_u^* = \langle S_i, \Lambda_i, g_i \rangle_{i\in I}$ which is mutually singular. Analogously to the case of $\mathcal{T}_u$, we call $\mathcal{T}_u^*$ the *canonical mutually singular lexicographic type structure*. In light of Proposition 3 and Proposition 4, both $\mathcal{T}_u$ and $\mathcal{T}_u^*$ satisfy a "richness" property, called completess (cf. [13]).

[13] Observe that some authors ([4], [29]) use the terminology "type space" for what is called "type structure" here.

Definition 8 *An $(S_i)_{i\in I}$-based lexicographic type structure $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i\in I}$ is **complete** if each belief map β_i is onto.*

Note that each type space in a complete lexicographic type structure has the cardinality of continuum. The structures $\mathcal{T}_u$ and $\mathcal{T}_u^*$ are particular instances of complete type structures. But there exist also complete type structures which are different from $\mathcal{T}_u$ and $\mathcal{T}_u^*$.[14]

3.5 From types to belief hierarchies

A lexicographic type structure provides an implicit representation about players' uncertainty, in the sense that it does not describe hierarchies of beliefs directly. In this Section we show that it is possible to associate with the subjective belief of each type an explicit hierarchy of beliefs. To accomplish this task, we fix a given $(S_i)_{i\in I}$-based type structure $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i\in I}$, and we define for each player $i \in I$ a *hierarchy description map* $d_i : T_i \to H_i^0$ associating with each $t_i \in T_i$ a corresponding hierarchy of LPS's. Following the terminology in [29], the hierarchy $d_i(t_i) = \left(d_i^1(t_i), d_i^2(t_i), ...\right)$ is called the *i-description* of t_i. Each hierarchy description map is defined inductively (cf. [4]):

- (base step: $k = 1$) For each $i \in I$, $t_i \in T_i$, define the first-order hierarchy description map $d_i^1 = \widehat{\text{Proj}}_{S_{-i}} \circ \beta_i : T_i \to \mathcal{N}(S_{-i})$ by
$$d_i^1(t_i) = \overline{\text{marg}}_{S_{-i}}(\beta_i(t_i)).$$
For each $i \in I$, let $\psi_{-i}^0 : S_{-i} \to S_{-i}$ be the identity map, and define $\psi_{-i}^1 : S_{-i} \times T_{-i} \to X_i^1 = S_{-i} \times \mathcal{N}(S_i)$ as
$$\psi_{-i}^1 = \left(Id_{S_{-i}}, d_{-i}^1\right).$$

- (inductive step: $k+1$, $k \geq 1$) Suppose we have already defined, for each $i \in I$, the functions $d_i^k : T_i \to \mathcal{N}\left(X_i^{k-1}\right)$ and $\psi_{-i}^k : S_{-i} \times T_{-i} \to X_i^k = X_i^{k-1} \times \mathcal{N}\left(X_{-i}^{k-1}\right)$. For each $i \in I$, $t_i \in T_i$, define $d_i^{k+1} : T_i \to \mathcal{N}\left(X_i^k\right)$ as
$$d_i^{k+1}(t_i) = \widehat{\psi}_{-i}^k(\beta_i(t_i));$$
consequently, the map $\psi_{-i}^{k+1} : S_{-i} \times T_{-i} \to X_i^{k+1}$ is defined as
$$\psi_{-i}^{k+1} = \left(\psi_{-i}^k, d_{-i}^{k+1}\right),$$
so that $\psi_{-i}^{k+1} = \left(s_{-i}, d_{-i}^1, ..., d_{-i}^k, d_{-i}^{k+1}\right)$.

It turns out that, for each $i \in I$, the map $\psi_{-i} : S_{-i} \times T_{-i} \to S_{-i} \times H_{-i}^0$ is given by $\psi_{-i} = \left(Id_{S_{-i}}, d_{-i}\right)$.

An easy check (use Lemma 4 in the base step, and then proceed by induction) shows that each d_i is a measurable function, and is continuous if each belief map is continuous. Consequently, the map $\widehat{\psi}_{-i} = \widehat{\left(Id_{S_{-i}}, d_{-i}\right)} : \mathcal{N}(S_{-i} \times T_{-i}) \to \mathcal{N}\left(S_{-i} \times H_{-i}^0\right)$ is continuous provided d_i is continuous.

[14] A simple but elegant argument was first used by BFK ([15, Proposition 7.2]) to state the existence of a belief-complete type structure $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i\in I}$ where each type space is Polish and each S_i is a finite, discrete space. Such an argument can be easily adapted to our framework as follows. Every Lusin space is analytic, so it is the image of the Baire space $\mathbb{N}^{\mathbb{N}}$ under a continuous map ([20, Corollary 8.2.8]; see also [34, p. 85])). For given spaces of primitive uncertainty $(S_i)_{i\in I}$, let $T_i = \mathbb{N}^{\mathbb{N}}$, for each $i \in I$. The above result implies the existence of continuous belief maps β_i from T_i onto $\mathcal{N}(S_{-i} \times T_{-i})$. These maps give us a complete lexicographic type structure.

3.6 Type morphisms and universality

In what follows, we let $T = \Pi_{i\in I} T_i$. If X and Y are topological spaces, we say that the map $f : X \to Y$ is **bimeasurable** if it is Borel measurable and, for each Borel set $B \subseteq X$, $f(B)$ is Borel in Y.

Definition 9 *Let $\mathcal{T} = \langle S_i, T_i, \beta_i\rangle_{i\in I}$ and $\mathcal{T}' = \langle S_i, T'_i, \beta'_i\rangle_{i\in I}$ be two $(S_i)_{i\in I}$-based lexicographic type structures. For each $i \in I$, let $\varphi_i : T_i \to T'_i$ be a measurable map such that*

$$\beta'_i \circ \varphi_i = \left(\widehat{Id_{S_{-i}}, \varphi_{-i}}\right) \circ \beta_i,$$

*where $\left(\widehat{Id_{S_{-i}}, \varphi_{-i}}\right) : \mathcal{N}(S_{-i} \times T_{-i}) \to \mathcal{N}\left(S_{-i} \times T'_{-i}\right)$ is the image LPS map under $\left(Id_{S_{-i}}, \varphi_{-i}\right) : S_{-i} \times T_{-i} \to S_{-i} \times T'_{-i}$. Then the function $(\varphi_i)_{i\in I} : T \to T'$ is called **type morphism (from $\mathcal{T}$ to $\mathcal{T}'$)**.*

*The morphism is called **bimeasurable** (resp. **type isomorphism**) if the map $(\varphi_i)_{i\in I}$ is bimeasurable (resp. an isomorphism). Say $\mathcal{T}$ and $\mathcal{T}'$ are **isomorphic** if there is a type isomorphism between them.*

The notion of type morphism captures the idea that a type structure $\mathcal{T}$ is "contained in" another type structure $\mathcal{T}'$ if $\mathcal{T}$ can be mapped into $\mathcal{T}'$ in a way which preserves the beliefs associated with types. Condition (2) in the definition of type morphism expresses consistency between the function $\varphi_i : T_i \to T'_i$ and the induced function $\left(\widehat{Id_{S_{-i}}, \varphi_{-i}}\right) : \mathcal{N}(S_{-i} \times T_{-i}) \to \mathcal{N}\left(S_{-i} \times T'_{-i}\right)$. That is, the following diagram commutes:

$$\begin{tikzcd}
T_i \arrow[r, "\beta_i"] \arrow[d, "\varphi_i"] & \mathcal{N}(S_{-i} \times T_{-i}) \arrow[d, "\left(\widehat{Id_{S_{-i}}, \varphi_{-i}}\right)"] \\
T'_i \arrow[r, "\beta'_i"] & \mathcal{N}(S_{-i} \times T'_{-i})
\end{tikzcd} \tag{3.3}$$

The notion of type morphism does not make any reference to hierarchies of LPS's. But, as one should expect, the important property of type morphisms is that they preserve the explicit description of lexicographic belief hierarchies.

Proposition 5 *Let $\mathcal{T} = \langle S_i, T_i, \beta_i\rangle_{i\in I}$ and $\mathcal{T}' = \langle S_i, T'_i, \beta'_i\rangle_{i\in I}$ be two $(S_i)_{i\in I}$-based lexicographic type structures. If $(\varphi_i)_{i\in I} : T \to T'$ is a type morphism from $\mathcal{T}$ to $\mathcal{T}'$, then $d_i(T_i) \subseteq d_i(T'_i)$ for each $i \in I$.*

In words, Proposition 5 states that if $\mathcal{T}$ can be embedded into $\mathcal{T}'$, then every $(S_i)_{i\in I}$-based belief hierarchy that is generated by some type in $\mathcal{T}$ is also generated by some type in $\mathcal{T}'$. This formalizes the idea of viewing type morphisms as a manner to relate types in one structure to types in a wider structure. Heifetz and Samet ([29, Proposition 5.1]) provide the above result for the case of standard type structures. Proposition 5 is indeed a straightforward generalization of Heifetz and Samet's result, and its proof is omitted, since it relies on standard arguments.[15]

But there is also another important, conceptual property of type morphism as we elaborate in Appendix 5.2. Every lexicographic type structure defines the set of belief hierarchies that are

[15] The statement of Proposition 5 can be rephrased by saying that every type morphism is also a *hierarchy morphism*, i.e., a map between type structures which preserves the hierarchies of beliefs associated with types.

allowed for each player. So, in a sense specified below, a lexicographic type structure represents a set of restrictions on players' hierarchies of beliefs that are "transparent", that is, not only the restrictions hold, but there is common full belief in those restrictions. This idea of "transparency" (referred to as "context" by BFK[16]) is captured by the notion of **self-evident event** in a type structure. Fix two $(S_i)_{i\in I}$-based lexicographic type structures, viz. $\mathcal{T}$ and $\mathcal{T}'$, and a bimeasurable type morphism $(\varphi_i)_{i\in I} : T \to T'$ between them. If $(\varphi_i)_{i\in I}$ is bimeasurable, then the set $S \times \Pi_{i\in I}\varphi_i(T_i)$ is a well defined event in $S \times T'$, and it is called self-evident in $\mathcal{T}'$.[17] Proposition 12 in Appendix 5.2 shows that (a) if $\mathcal{T}$ is mapped via type morphism into $\mathcal{T}'$, then $\mathcal{T}$ corresponds to a self-evident event in $\mathcal{T}'$; and (b) every self-evident event in $\mathcal{T}$ corresponds to a "smaller" type structure.

Put differently, such result says that, if $\mathcal{T}$ can be mapped into $\mathcal{T}'$ by a (bimeasurable) type morphism $(\varphi_i)_{i\in I}$, we can essentially regard $\mathcal{T}$ as a (measurable) substructure of $\mathcal{T}'$. This raises the following question: Is there a lexicographic type structure into which any other type structure can be mapped? Alternatively put, since a lexicographic type structure generates hierarchies of LPS's, does there exists a type structure that generates all hierarchies of beliefs? A type structure satisfying this requirement is called universal.

Definition 10 *An $(S_i)_{i\in I}$-based type structure $\mathcal{T}' = \langle S_i, T_i', \beta_i'\rangle_{i\in I}$ is **universal** if for every other $(S_i)_{i\in I}$-based type structure $\mathcal{T} = \langle S_i, T_i, \beta_i\rangle_{i\in I}$ there is a unique type morphism from $\mathcal{T}'$ to $\mathcal{T}$. In this case, the set $S \times T'$ is called **universal belief space**.*

Of course, any two universal type structures are isomorphic.

We state now the main result of this section.

Theorem 1 *Let $\mathcal{T} = \langle S_i, T_i, \beta_i\rangle_{i\in I}$ be an arbitrary $(S_i)_{i\in I}$-based lexicographic type structure, and, for each $i \in I$, let $d_i : T_i \to H_i^0$ be an i-description map. Then, for each $i \in I$,*

1. *$d_i(T_i) \subseteq H_i$,*
2. *$(d_i)_{i\in I}$ is the unique type morphism from $\mathcal{T}$ to $\mathcal{T}_u = \langle S_i, H_i, \overline{f}_i\rangle_{i\in I}$.*

   Thus $\mathcal{T}_u$ is the unique universal lexicographic type structure (up to type isomorphism).

Note that the type structure $\mathcal{T}$ in Theorem 1 does not necessarily give rise to a self-evident event in $\mathcal{T}_u$. This is so because the type morphism $(d_i)_{i\in I}$ from $\mathcal{T}$ to $\mathcal{T}_u$ may fail to be bimeasurable.[18] We now provide sufficient conditions on type structures under which the requirement of bimeasurability is satisfied.

[16] As BFK put it ([15, p.319]), a specific lexicographic type structure can be thought of as "... giving the "context" in which the game is played", so that "... who the players are in the given game can be seen as a shorthand for their experiences before the game. The players' possible characteristics—including their possible types—then reflect the prior history or context."

[17] By bimeasurability of φ_i, the set $\varphi_i(T_i)$ is a Lusin subspace of T_i', hence Borel in T_i'. The product space $S \times \Pi_{i\in I}\varphi_i(T_i)$ is sometimes called *belief-closed subspace* of $S \times T'$ (cf. [14, Remark 2] and [50]). Here, we refrain from using such terminology since the original definition of belief-closed subspace, due to Mertens and Zamir [39], is stated within the formalism of *belief spaces* and *belief morphisms*. Both definitions of belief spaces and belief morphisms are more comprehensive than those of type structures and type morphisms, respectively. But, as remarked by Heifetz and Samet ([29, Section 6]), they do not give rise to different definitions of epistemic types.

[18] The bimeasurability condition for type morphisms is automatically satisfied in Mertens and Zamir's framework (cf. [39]), since all the spaces are compact and all the relevant functions are continuous.

Definition 11 *Call a lexicographic type structure $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i \in I}$* ***countable*** *(resp.* ***finite****) if the cardinality of each type space T_i is countable (resp. finite).*

We recall that each finite or countable set is endowed with the discrete topology (which makes it a Polish space), so the above definition of finite (resp. countable) type structure is well-posed. We also introduce an important class of type structures, namely type structures satisfying a *non-redundancy* condition. A type structure is non-redundant if any two distinct types induce distinct lexicographic belief hierarchies. Formally:

Definition 12 *An $(S_i)_{i \in I}$-based lexicographic type structure $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i \in I}$ is* ***non-redundant*** *if, for each $i \in I$, the i-description map d_i is injective.*[19] *Say $\mathcal{T}$ is* ***redundant*** *if it is not non-redundant.*

It is evident from this definition that both $\mathcal{T}_u$ and $\mathcal{T}_u^*$ are non-redundant, as each i-description map turns out to be an isomorphism. The following result (see Appendix 5.2 for the proof) shows that, for countable and\or non-redundant type structures, the bimeasurability problem for $(d_i)_{i \in I}$ is avoided.[20]

Proposition 6 *If $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i \in I}$ is countable or non-redundant lexicographic type structure, then $S \times \Pi_{i \in I} d_i (T_i)$ is a self-evident event in $\mathcal{T}_u$. Conversely, for each self-evident event $S \times \Pi_{i \in I} E_i \subseteq S \times H$ in $\mathcal{T}_u$, there exists a non-redundant type structure $\mathcal{T}' = \langle S_i, T_i', \beta_i' \rangle_{i \in I}$ such that $(d_i)_{i \in I} : T' \to \Pi_{i \in I} E_i$ is a type isomorphism.*

3.7 Mutually singular type structures and universality

Note that Theorem 1 identifies the structure $\mathcal{T}_u = \langle S_i, H_i, \overline{f}_i \rangle_{i \in I}$ as the terminal object in the category of all possible type structures, i.e., type structures where LPS's are not required to be mutually singular. This raises the following question: Is there a universal structure within the class of mutually singular type structures? One would expect the structure $\mathcal{T}_u^* = \langle S_i, \Lambda_i, g_i \rangle_{i \in I}$ to be the natural candidate for this class of type structures. But the following example shows $\mathcal{T}_u^*$ could not work for a very simple reason: A mutually singular type may induce a hierarchy of beliefs which is not mutually singular *and* cannot be represented by a mutually singular LPS over opponent's strategies and hierarchies.

Example 1 *Consider the following game, where $S_1 = \{U, M, D\}$ and $S_2 = \{L, C, R\}$.*

$1\backslash 2$	L	C	R
U	$(4,1)$	$(4,1)$	$(0,1)$
M	$(0,1)$	$(0,1)$	$(4,1)$
D	$(3,1)$	$(2,1)$	$(2,1)$

[19] Mertens and Zamir ([39, Definition 2.4 and Proposition 2.5]) formulate the non-redundancy condition in terms of a separation condition which is equivalent the property stated here. According to their formulation, a type structure $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i \in I}$ is non-redundant if the σ-field on each T_i generated by d_i separates the points. It is shown in [38] that both definitions are equivalent within the framework of standard type structures. The extension of this result to the case of lexicographic type structures is straightforward.

[20] Different conditions, weaker than countability and non-redundancy, can be imposed on lexicographic type structures to guarantee the bimeasurability property of a type morphism (see [3, Appendix A] for details). Countability and non-redundancy suffice for the purposes of the present paper.

We append to this game the following mutually singular type structure $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i \in I}$. For the set of types, take $T_1 = \{t_1'\}$ and $T_2 = \{t_2', t_2''\}$. The belief maps $\beta_1 : T_1 \to \mathcal{L}(S_2 \times T_2)$ and $\beta_2 : T_2 \to \mathcal{L}(S_1 \times T_1)$ are defined as follows. Player 1's type t_1' is associated with a length-2 LPS $\beta_1(t_1') = (\nu_1^1, \nu_1^2)$, such that

$$\begin{aligned} \nu_1^1(\{C\} \times \{t_2'\}) &= \nu_1^1(\{R\} \times \{t_2'\}) = \frac{1}{2}, \\ \nu_1^2(\{L\} \times \{t_2''\}) &= \nu_1^2(\{R\} \times \{t_2''\}) = \frac{1}{2}. \end{aligned}$$

Player 2's belief map is such that $\beta_2(t_2') = \beta_2(t_2'')$, a mutually singular LPS. It is easily verified that the LPS $\beta_1(t_1') = (\nu_1^1, \nu_1^2)$ is mutually singular—specifically, ν_1^1 and ν_1^2 have disjoint supports, given by $\mathrm{Supp}\nu_1^1 = \{C, R\} \times \{t_2'\}$ and $\mathrm{Supp}\nu_1^2 = \{L, R\} \times \{t_2''\}$, respectively. However, the induced first-order belief $\overline{\mathrm{marg}}_{S_2}(\beta_1(t_1')) = (\mathrm{marg}_{S_2}\nu_1^1, \mathrm{marg}_{S_2}\nu_1^2)$ is not mutually singular—the supports of the marginal probability measures are given by the sets $\{C, R\}$ and $\{L, R\}$, respectively. Moreover, $\mathrm{marg}_{T_2}\nu_1^1$ and $\mathrm{marg}_{T_2}\nu_1^2$ assign probability 1 respectively on t_2' and t_2'', which obviously induce the same hierarchy of player 2. The two things together imply that (i) all induced higher-order beliefs are not mutually singular (formal proof in Appendix 5.3), and that (ii) the induced hierarchy can be represented only by a non mutually singular, length-2 LPS over strategy-hierarchy pairs of the opponent, where both component measures assign probability $1/2$ to the same pair.

Example 1 shows two different, but related difficulties concerning the notion of mutual singularity for lexicographic type structures. The first difficulty is, in some sense, operational: The type structure $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i \in I}$ described in Example 1 is simple enough to conclude by a simple induction argument (see Appendix 5.3) that the hierarchy $d_1(t_1')$ is not mutually singular at *any* order and cannot be represented by a mutually singular LPS over strategy-hierarchy pairs of the opponent. But, for more "complicated" type structure, doing these checks could be a very difficult task.

The second difficulty is instead conceptual: Is there a (sub)class of mutually singular type structures such that $\mathcal{T}_u^* = \langle S_i, \Lambda_i, g_i \rangle_{i \in I}$ is the universal structure *within* this class? If the answer is affirmative, then what modeling assumptions are captured by the mutual singularity condition on a type structure? How do those assumptions relate to the notion of mutually singular hierarchies?

We overcome such difficulties by providing a strengthening of the notion of mutual singularity, called **strong mutual singularity**, which is defined within the (lexicographic) type structure formalism, without any reference to hierarchies of beliefs. This notion, which is of measure-theoretic nature, solves the aforementioned problems (both conceptual and operational), and builds on the important work of Friedenberg and Meier [28] concerning the relationship between hierarchies and type morphisms.

We begin our analysis with a measurability condition concerning the belief maps of a type structure, following Friedenberg and Meier [28]:

Definition 13 *Fix a type structure $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i \in I}$, and, for each $i \in I$, sub-σ-fields $\mathcal{F}_{T_i} \subseteq \Sigma_{T_i}$. Say that $\Pi_{i \in I}\mathcal{F}_{T_i}$ is **closed under** $\mathcal{T}$ if, for each $i \in I$, $E_{-i} \times F_{-i} \in \Sigma_{S_{-i}} \times \mathcal{F}_{T_{-i}}$, $(p_1, p_2, ..., p_n) \in (\mathbb{Q} \cap [0,1])^n$, it holds that*

$$(\beta_i)^{-1}\left\{(\mu_i^1, ..., \mu_i^n) \in \mathcal{N}(S_{-i} \times T_{-i}) \,\middle|\, \mu_i^l(E_{-i} \times F_{-i}) \geq p_l, \forall l \leq n\right\} \in \mathcal{F}_{T_i}.$$

For a given type structure $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i \in I}$, let $\left\{ \Pi_{i \in I} \mathcal{F}^{\theta}_{T_i} \right\}_{\theta \in \Theta}$ be the family of all sub-σ-fields closed under $\mathcal{T}$. For each $i \in I$, define $\mathcal{G}_{T_i} = \cap_{\theta \in \Theta} \mathcal{F}^{\theta}_{T_i}$. Clearly, $\Pi_{i \in I} \mathcal{G}_{T_i}$ is a sub-σ-field of $\Pi_{i \in I} \mathcal{F}_{T_i}$, and it is closed under $\mathcal{T}$. So $\Pi_{i \in I} \mathcal{G}_{T_i}$ is called the **coarsest σ-field closed under $\mathcal{T}$**.

Referring back to Example 1, note that there are two σ-fields closed under $\mathcal{T}$, namely $\{\emptyset, T_1\} \times \{\emptyset, T_2, \{t'_2\}, \{t''_2\}\}$ and $\{\emptyset, T_1\} \times \{\emptyset, T_2\}$. The latter is the coarsest σ-field closed under $\mathcal{T}$.

Next result extends [28, Proposition 5.1] to the present framework.

Proposition 7 *For a given type structure $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i \in I}$ it holds that*

$$\Pi_{i \in I} \mathcal{G}_{T_i} = \Pi_{i \in I} \sigma(d_i).$$

The notion of $\Pi_{i \in I} \mathcal{G}_{T_i}$ is defined within the domain of lexicographic type structures, so this leaves open the question as to how to interpret the condition. Proposition 7 above establishes that the coarsest σ-field closed under $\mathcal{T}$ is precisely the σ-field generated by the hierarchy description maps. So substantially $\Pi_{i \in I} \mathcal{G}_{T_i}$ defines a sub-language of type spaces which corresponds to the players' language in the hierarchy space.

Given $\overline{\mu} = (\mu_1, ..., \mu_n) \in \mathcal{N}(X)$ and a sub-σ-field $\mathcal{F}_X \subseteq \Sigma_X$, say $\overline{\mu}$ is a **mutually singular w.r.to** $\mathcal{F}_X$ if, for each $l = 1, ..., n$, there are sets $E_l \in \mathcal{F}_X$ such that $\mu_l(E_l) = 1$ and $\mu_l(E_m) = 0$ for $l \neq m$.

Proposition 8 *Fix a mutually singular type structure $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i \in I}$. A type $t_i \in T_i$ induces a hierarchy $d_i(t_i) \in \Lambda^1_i$ if and only if $\beta_i(t_i)$ is mutually singular w.r.to $\Sigma_{S_{-i}} \times \mathcal{G}_{T_{-i}}$.*

The above result provides an operationally convenient way to check whether a mutually singular type, viz. $t_i \in T_i$, induces or not a hierarchy which is has a mutually singular representation. The notion of coarsest σ-field closed under $\mathcal{T}$ is defined on the type structure alone. So, in order to check the mutually singular representation of a hierarchy induced by the type t_i, there is no need to leave the domain of type structures—we simply need to check that $\beta_i(t_i)$ is mutually singular w.r.to $\Sigma_{S_{-i}} \times \mathcal{G}_{T_{-i}}$. To see the significance of this, refer back to Example 1: Player 1's type t'_1 cannot induce a hierarchy with a mutually singular representation, in that the corresponding LPS $\beta_1(t'_1) = (\nu^1_1, \nu^2_1)$ is not mutually singular w.r.to $\Sigma_{S_2} \times \mathcal{G}_{T_2} = 2^{S_2} \times \{\emptyset, T_2\}$. Note that Proposition 8 is automatically satisfied in the mutually singular canonical type structure $\mathcal{T}^*_u = \langle S_i, \Lambda_i, g_i \rangle_{i \in I}$.

With this in place, we can now introduce an important class of lexicographic type structures.

Definition 14 *Let $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i \in I}$ be a mutually singular type structure. Say $\mathcal{T}$ is* ***strongly mutually singular*** *if, for each player $i \in I$, each type $t_i \in T_i$ is such that $\beta_i(t_i)$ is mutually singular w.r.to $\Sigma_{S_{-i}} \times \mathcal{G}_{T_{-i}}$.*

Note that $\mathcal{T}^*_u = \langle S_i, \Lambda_i, g_i \rangle_{i \in I}$ is strongly mutually singular. We can now state the main result concerning the class of strongly mutually singular type structures.

Theorem 2 *Let $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i \in I}$ be an arbitrary strongly mutually singular type structure, and, for each $i \in I$, let $d_i : T_i \to H^0_i$ be the i-description map. Then, for each $i \in I$,*

1. $d_i(T_i) \subseteq \Lambda_i$,

2. $(d_i)_{i\in I}$ *is the unique type morphism from* $\mathcal{T}$ *to* $\mathcal{T}_u^* = \langle S_i, \Lambda_i, g_i \rangle_{i\in I}$.

   Thus $\mathcal{T}_u^*$ *is the unique universal lexicographic type structure (up to type isomorphism) within the class of strongly mutually singular type structures.*

We next provide an interesting case to check whether a type structure is strongly mutually singular. We have already introduced the concept of non-redundant type structure (Definition 12).[21] Now the claim is:

Proposition 9 *Let* $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i\in I}$ *be a mutually singular type structure. If* $\mathcal{T}$ *is non-redundant, then* $\mathcal{T}$ *is strongly mutually singular.*

Proof: First note that, if $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i\in I}$ is non-redundant, then each belief map $\beta_i : T_i \to \mathcal{L}(S_{-i} \times T_{-i})$ is injective, hence a measure-theoretic embedding by Souslin Theorem. To see this, observe that the map $(d_i)_{i\in I}$ is bimeasurable by Proposition 6, so that also the map $\widehat{\psi}_{-i} = \left(\widehat{Id_{S_{-i}}, d_{-i}}\right)$ is bimeasurable. By Theorem 1, the following diagram commutes:

$$\begin{tikzcd}
T_i \arrow[r, "\beta_i"] \arrow[d, "d_i"] & \mathcal{L}(S_{-i} \times T_{-i}) \arrow[d, "(\widehat{Id_{S_{-i}}, d_{-i}})"] \\
H_i \arrow[r, "\overline{f}_i"] & \mathcal{N}(S_{-i} \times H_{-i})
\end{tikzcd}$$

As such, each belief map β_i is bimeasurable. Now, pick any $t_i \in T_i$. Since $\widehat{\psi}_{-i}$ is bimeasurable, then $\widehat{\psi}_{-i}(\beta_i(t_i))$ is a mutually singular LPS over $S_{-i} \times H_{-i}$. But $\widehat{\psi}_{-i}(\beta_i(t_i)) = \overline{f}_i(d_i(t_i))$, so Corollary **??** implies that $d_i(t_i) \in \Lambda_i^1$. It follows from Proposition 8 that $\beta_i(t_i)$ is mutually singular w.r.to $\Sigma_{S_{-i}} \times \mathcal{G}_{T_{-i}}$. Since $t_i \in T_i$ is arbitrary, the conclusion follows. ■

Note that the canonical type structure $\mathcal{T}_u^*$ is non-redundant, so the result in Proposition 9 holds automatically.

We conclude this section with an example which shows how the issue of non-redundancy characterizes strongly mutually type structures.

Example 2 *We consider two variants of Example 1. In the first case we show that a redundant type structure* $\mathcal{T}$ *can be strongly mutually singular, which shows that Proposition 9 above provides a sufficient, but not necessary, condition for* $\mathcal{T}$ *to be strongly mutually singular. In the second variant, we provide an example where Proposition 9 above holds.*

For the first case: Suppose that the belief map $\beta_1 : T_1 \to \mathcal{L}(S_2 \times T_2)$ *is defined as follows: Player 1's type* t_1' *is associated with a length-2 LPS* $\beta_1(t_1') = (\nu_1^1, \nu_1^2)$*, so that*

$$\begin{aligned}
\nu_1^1\left(\{C\} \times \{t_2'\}\right) &= \nu_1^1\left(\{R\} \times \{t_2'\}\right) = \frac{1}{2}, \\
\nu_1^2\left(\{L\} \times \{t_2''\}\right) &= 1.
\end{aligned}$$

[21] Given a measurable space (X, Σ_X), say that $\mathcal{B} \subseteq \Sigma_X$ is **separated** if for each $x, x' \in X$ there is $B \in \mathcal{B}$ such that $x \in B$ and $x' \notin B$. In the context of standard type structures, Friedenberg and Meier show that a type structure $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i\in I}$ is non-redundant if and only if, for each $i \in I$, $\mathcal{G}_{T_i}$ is separated ([28, Lemma 7.2]). Of course, such result holds true even if $\mathcal{T}$ is a lexicographic type structure.

Clearly the LPS $\beta_1(t_1') = (\nu_1^1, \nu_1^2)$ is mutually singular, since ν_1^1 and ν_1^2 have disjoint supports, given by $\mathrm{Supp}\nu_1^1 = \{C, R\} \times \{t_2'\}$ *and* $\mathrm{Supp}\nu_1^1 = \{L\} \times \{t_2''\}$, *respectively. Furthermore, the induced first-order belief* $\overline{\mathrm{marg}}_{S_2}(\beta_1(t_1')) = (\mathrm{marg}_{S_2}\nu_1^1, \mathrm{marg}_{S_2}\nu_1^2)$ *is also mutually singular - the supports of the marginal probability measures are the sets $\{C, R\}$ and $\{L\}$, respectively. So, the type structure is strongly mutually singular, despite the fact that it is redundant. Note also that this new LPS $\beta_1(t_1') = (\nu_1^1, \nu_1^2)$ is mutually singular w.r.to $\Sigma_{S_2} \times \mathcal{G}_{T_2} = 2^{S_2} \times \{\emptyset, T_2\}$. (The sets $\{C, R\} \times T_2$ and $\{L\} \times T_2$ are disjoint and satisfy the required properties.)*

For the second case: Suppose now that Player 2's belief map is such that $\beta_2(t_2') \neq \beta_2(t_2'')$, both mutually singular, and $\beta_1(t_1')$ is as in Example 1. So the type structure $\mathcal{T}$ is non-redundant. The LPS $\beta_1(t_1') = (\nu_1^1, \nu_1^2)$ is mutually singular w.r.to $\Sigma_{S_2} \times \mathcal{G}_{T_2} = 2^{S_2} \times \{\emptyset, T_2, \{t_2'\}, \{t_2''\}\}$ - indeed, the sets $\mathrm{Supp}\nu_1^1 = \{C, R\} \times \{t_2'\}$ *and* $\mathrm{Supp}\nu_1^2 = \{L, R\} \times \{t_2''\}$ *are disjoint and satisfy the required properties.*

4 Terminal type structures

Besides completeness, the literature on epistemic game theory have provided a related notion of "large" type structures, namely (**finitely**) **terminal type structures**. In the definition below, fix two $(S_i)_{i\in I}$-based lexicographic type structures, namely $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i\in I}$ and $\mathcal{T}' = \langle S_i, T_i', \beta_i' \rangle_{i\in I}$.

Definition 15 *An $(S_i)_{i\in I}$-based type structure $\mathcal{T}$ is **finitely terminal** if, for each $(S_i)_{i\in I}$-based type structure $\mathcal{T}'$, each type $t_i' \in T_i'$ and each $k \in \mathbb{N}$, there is a type $t_i \in T_i$ such that*

$$\left(d_i^1(t_i), ..., d_i^k(t_i)\right) = \left(d_i^1(t_i'), ..., d_i^k(t_i')\right).$$

Definition 16 *An $(S_i)_{i\in I}$-based type structure $\mathcal{T}$ is **terminal** if, for each $(S_i)_{i\in I}$-based type structure $\mathcal{T}'$ and each type $t_i' \in T_i'$, there is a type $t_i \in T_i$ with $d_i(t_i') = d_i(t_i)$.*

Definition 15 and Definition 16 are due to Friedenberg [27]. Clearly, the canonical type structure $\mathcal{T}_u = \langle S_i, H_i, \overline{f}_i \rangle_{i\in I}$ is terminal (and, of course, also finitely terminal). This section addresses the following question: Is a complete lexicographic type structure terminal? In the context of ordinary probabilities (i.e., Subjective Expected Utility preferences) Friedenberg ([27, Theorem 3.1]) shows that a complete type structure is terminal provided each type space is compact and each belief map is continuous. In the lexicographic case, however, there is no analogue of the aforementioned result. Yet we provide a limited statement: A complete lexicographic type structure is *finitely* terminal. We do this by showing a stronger result: A type structure which induces all mutually singular LPS's is finitely terminal. As a by-product, this entails that the canonical mutually singular type structure $\mathcal{T}_u^* = \langle S_i, \Lambda_i, g_i \rangle_{i\in I}$ is finitely terminal.

In what follows, for each player $i \in I$, let $\overline{H}_i^k$ denote the of k-order belief hierarchies consistent with common full belief in coherence; that is,

$$\overline{H}_i^k = \left\{ \left(\overline{\mu}_i^1, \overline{\mu}_i^2, ..., \overline{\mu}_i^k\right) \in \Pi_{l=0}^{k-1}\mathcal{N}\left(X_{-i}^l\right) \;\middle|\; \begin{array}{c} \exists h_i \in H_i, \\ \mathrm{Proj}_{\Pi_{l=0}^{k-1}\mathcal{N}\left(X_{-i}^l\right)}(h_i) = \left(\overline{\mu}_i^1, \overline{\mu}_i^2, ..., \overline{\mu}_i^k\right) \end{array} \right\}.$$

So, for each player $i \in I$, we define a *k-order hierarchy description map* $\widetilde{d}_i^k : T_i \to \Pi_{l=0}^{k-1} \mathcal{N}\left(X_{-i}^l\right)$ as follows: For all $k \in \mathbb{N}$,

$$\widetilde{d}_i^k(t_i) = \left(d_i^1(t_i), d_i^2(t_i), ..., d_i^k(t_i)\right), \ t_i \in T_i.$$

Remark 1 *For all $k \in \mathbb{N}$, it holds that*

$$\widetilde{d}_i^k(t_i) \in \overline{H}_i^k, \ \forall t_i \in T_i.$$

To see this, note that

$$\widetilde{d}_i^k(t_i) = \mathrm{Proj}_{\Pi_{l=0}^{k-1}\mathcal{N}\left(X_{-i}^l\right)}(d_i(t_i)), \ \forall t_i \in T_i,$$

and $d_i(t_i) \in H_i$ by Theorem 1.

It is also noteworthy that, for all $k \in \mathbb{N}$,

$$\widetilde{d}_i^k(t_i) = \left(\widetilde{d}_i^{k-1}(t_i), d_i^k(t_i)\right), \ t_i \in T_i.$$

Let us reformulate Definitions 15 and 16 in a more compact way:

Remark 2 *An $(S_i)_{i\in I}$-based type structure $\mathcal{T}$ is **finitely terminal** if, for each $(S_i)_{i\in I}$-based type structure $\mathcal{T}'$, we have*

$$\widetilde{d}_i^k(T_i') \subseteq \widetilde{d}_i^k(T_i), \ \forall k \in \mathbb{N}, \ \forall i \in I.$$

*An $(S_i)_{i\in I}$-based type structure $\mathcal{T}$ is **terminal** if, for each $(S_i)_{i\in I}$-based type structure $\mathcal{T}'$, we have*[22]

$$d_i(T_i') \subseteq d_i(T_i), \ \forall i \in I.$$

The following result establishes the relationship between any (finitely) terminal type structure and $\mathcal{T}_u$.

Proposition 10 *Fix an $(S_i)_{i\in I}$-based lexicographic type structure $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i\in I}$.*

(i) *$\mathcal{T}$ is finitely terminal if and only if $\widetilde{d}_i^k(T_i) = \overline{H}_i^k$ for each $k \in \mathbb{N}$ and each $i \in I$.*

(ii) *$\mathcal{T}$ is terminal if and only if $d_i(T_i) = H_i$ for each $i \in I$.*

Proof: We prove only part (ii) (the proof of Part (i) is virtually identical). Since $d_i(T_i) \subseteq H_i$, we need to show that $H_i \subseteq d_i(T_i)$. We shall make use of the characterization of the notion of terminality given in Remark 2.

If $\mathcal{T}$ is terminal, then for the structure $\mathcal{T}_u$ it holds that

$$d_i(H_i) = H_i \subseteq d_i(T_i).$$

[22] Cf. Remark 3.1 in Friedenberg and Meier [28] concerning the definition of hierarchy morphism.

Conversely, let $\mathcal{T}$ be such that $d_i(T_i) = H_i$. For every other type structure $\mathcal{T}' = \langle S_i, T_i', \beta_i' \rangle_{i \in I}$ it holds that $d_i(T_i') \subseteq H_i = d_i(T_i)$, so $\mathcal{T}$ is terminal. ■

Thus, Proposition 10 provides a useful characterization of the (finite) terminality property of each type structure we shall be using in the proof of the main result. It is basically a version of Result 2.1 (and Proposition B1.(ii)) in Friedenberg [27]. Of course, if $\mathcal{T}$ is terminal, then it is also finitely terminal.

In order to provide the main result of this section, we need an additional definition. We say that a type structure $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i \in I}$ is **ms-complete** if $\beta_i(T_i) \supseteq \mathcal{L}(S_{-i} \times T_{-i})$ for each $i \in I$.

Theorem 3 *Fix an $(S_i)_{i \in I}$-based lexicographic type structure $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i \in I}$. Thus, if $\mathcal{T}$ is ms-complete, then $\mathcal{T}$ is finitely terminal.*

Proof: For each $i \in I$ and $k \geq 1$, define the map $\overline{d}_i^k : \left(d_i^k\right)^{-1}\left(\mathcal{L}\left(X_i^{k-1}\right)\right) \to \mathcal{L}\left(X_i^{k-1}\right)$ as $\overline{d}_i^k(t_i) = d_i^k(t_i)$ for all $t_i \in \left(d_i^k\right)^{-1}\left(\mathcal{L}\left(X_i^{k-1}\right)\right)$. We first show the following fact: If $\mathcal{T}$ is ms-complete, then, for all $k \geq 1$, the map $\overline{d}_i^k$ is onto, for each player $i \in I$. This is true for $k = 1$: By Lemma 5.(3), for each $\overline{\mu}_i^1 \in \mathcal{L}\left(X_i^0\right) = \mathcal{L}(S_{-i})$ there exists $\overline{\nu}_i \in \mathcal{L}(S_{-i} \times T_{-i})$ such that $\widehat{\mathrm{Proj}}_{S_{-i}}(\overline{\nu}_i) = \overline{\mu}_i^1$. By ms-completeness, there exists $t_i^{(1)} \in T_i$ such that $\beta_i\left(t_i^{(1)}\right) = \overline{\nu}_i$. Hence, $d_i^1(t_i^{(1)}) = \overline{\mu}_i^1$. This shows that $\overline{d}_i^1$ is onto. Hence also the Borel map $\overline{\psi}_{-i}^1 = \left(Id_{S_{-i}}, \overline{d}_{-i}^1\right)$ is onto. Suppose by way of induction that the statement is true for $k \geq 1$ and we have defined the Borel surjective map $\overline{\psi}_{-i}^k = \left(\overline{\psi}_{-i}^{k-1}, \overline{d}_{-i}^k\right) : S_{-i} \times T_{-i} \to X_i^k$. By Lemma 5.(3), for each $\overline{\mu}_i^{k+1} \in \mathcal{L}\left(X_i^k\right)$ there exists $\overline{\nu}_i \in \mathcal{L}(S_{-i} \times T_{-i})$ such that $\widehat{\overline{\psi}_{-i}^k}(\overline{\nu}_i) = \overline{\mu}_i^{k+1}$. By ms-completeness, there exists $t_i^{(k+1)} \in T_i$ such that $\beta_i\left(t_i^{(k+1)}\right) = \overline{\nu}_i$. Hence, $d_i^k(t_i^{(k+1)}) = \overline{\mu}_i^{k+1}$, and this shows that $\overline{d}_i^{k+1}$ is onto. Hence, also the Borel map $\overline{\psi}_{-i}^{k+1} = \left(\overline{\psi}_{-i}^k, \overline{d}_{-i}^{k+1}\right) : S_{-i} \times T_{-i} \to X_i^{k+1}$ is surjective.

We now show that, for all $k \geq 1$, the map $\widetilde{d}_i^k : T_i \to \overline{H}_i^k$ is onto for each $i \in I$. By Proposition 10, it will follow that $\mathcal{T}$ is finitely terminal, as required. Fix $k \geq 1$ and pick any $\left(\overline{\mu}_i^1, \overline{\mu}_i^2, ..., \overline{\mu}_i^k\right) \in \overline{H}_i^k$. Let $\left(\overline{\nu}_i^1, \overline{\nu}_i^2, ..., \overline{\nu}_i^k, \overline{\nu}_i^{k+1}\right) \in \overline{H}_i^{k+1}$ such that $\overline{\nu}_i^l = \overline{\mu}_i^l$ for all $l \leq k$ and $\overline{\nu}_i^{k+1} \in \mathcal{L}\left(X_i^k\right)$. In view of the above, the map $\overline{d}_i^{k+1}$ is onto, so there exists $t_i^{(k+1)} \in T_i$ such that $\overline{d}_i^{k+1}\left(t_i^{(k+1)}\right) = d_i^{k+1}\left(t_i^{(k+1)}\right) = \overline{\nu}_i^{k+1}$. We need to show that

$$\widetilde{d}_i^k\left(t_i^{(k+1)}\right) = \left(\overline{\nu}_i^1, \overline{\nu}_i^2, ..., \overline{\nu}_i^k\right) = \left(\overline{\mu}_i^1, \overline{\mu}_i^2, ..., \overline{\mu}_i^k\right).$$

Fix $l \geq 1$ with $l \leq k$. By coherence of the induced hierarchies, it follows that

$$\overline{\mathrm{marg}}_{X_i^{l-1}}\left(d_i^{k+1}\left(t_i^{(k+1)}\right)\right) = d_i^l\left(t_i^{(k+1)}\right).$$

But $\left(\overline{\nu}_i^1, \overline{\nu}_i^2, ..., \overline{\nu}_i^k, \overline{\nu}_i^{k+1}\right)$ is also coherent, so

$$\overline{\mathrm{marg}}_{X_i^{l-1}}\overline{\nu}_i^{k+1} = \overline{\nu}_i^l.$$

Since $d_i^{k+1}\left(t_i^{(k+1)}\right) = \overline{\nu}_i^{k+1}$, we can conclude that

$$d_i^l\left(t_i^{(k+1)}\right) = \overline{\nu}_i^l.$$

■

Since a complete type structure is ms-complete, the main result of this section immediately follows from Theorem 3.

Corollary 1 *Fix an $(S_i)_{i\in I}$-based lexicographic type structure $\mathcal{T} = \langle S_i, T_i, \beta_i\rangle_{i\in I}$. Thus, if $\mathcal{T}$ is complete, then $\mathcal{T}$ is finitely terminal.*

Denote by $T_i[h_i|k]$ the set of types in T_i whose induced hierarchy of lexicographic beliefs agree with $h_i = \left(\overline{\mu}_i^1, \overline{\mu}_i^2, ...\right)$ at level k. Clearly the sequence $\{T_i[h_i|k]\}_{k\geq 1}$ is *non-increasing*, and $\cap_{k\geq 1} T_i[h_i|k]$ represents the sets of types whose induced hierarchy of beliefs agree with h_i. This raises the question as to whether $\cap_{k\geq 1} T_i[h_i|k] \neq \emptyset$. If we impose the requirement of continuity of each belief map β_i, then each k-order hierarchy description map $\widetilde{d}_i^k$ is continuous (by Lemma 4), and $T_i[h_i|k]$ is the continuous inverse image of the singleton $\left(\mu_i^1, ..., \mu_i^k\right) \in \overline{H}_i^k$, i.e., a closed subset of T_i. If each T_i were compact, then an analogous conclusion would hold for $T_i[h_i|k]$, so that $\cap_{k\geq 1} T_i[h_i|k] \neq \emptyset$ by the finite intersection property. This would imply the existence of $t_i \in \cap_{k\geq 1} T_i[h_i|k] \subseteq T_i$ such that $d_i(t_i) = h_i$, i.e., $\widetilde{h}_i(T_i) = \overline{H}_i$ for each $i \in I$. However, a complete type structure with continuous belief maps and compact type spaces does not exist in this setting. As such, the conclusion of the Corollary 1 appears to be tight.

5 Appendix

5.1 Proofs for Section 3

5.1.1 Properties of image LPS maps

We first report an auxiliary technical fact we shall be using in the proofs that follow.

Lemma 3 *Let $\{f_n\}_{n\in\mathbb{N}}$ be a countable family of mappings between topological spaces, where $f_n : X_n \to Y$. Thus if each map f_n is continuous (resp. Borel measurable, open), then $\cup_{n\in\mathbb{N}} f_n : X \to Y$ is continuous (resp. Borel measurable, open).*

Proof: Let O be open in Y. Thus

$$\left(\cup_{n\in\mathbb{N}} f_n\right)^{-1}(O) = \cup_{n\in\mathbb{N}} f_n^{-1}(O).$$

Therefore, if each f_n is continuous (resp. Borel measurable), then each $f_n^{-1}(O)$ is open (resp. Borel), which in turn implies that $\left(\cup_{n\in\mathbb{N}} f_n\right)^{-1}(O)$ is open (resp. Borel). If U is open in X, then

$$\left(\cup_{n\in\mathbb{N}} f_n\right)(U) = \cup_{n\in\mathbb{N}} f_n(U).$$

So, if each f_n is open, then $\cup_{n\in\mathbb{N}} f_n(U)$ is open in Y, establishing the result. ■

Remark 3 *For the continuous and open cases, the result remains true for an arbitrary (not necessarily countable) family of mappings.*

Lemma 4 *Let X and Y be metrizable Lusin spaces, and fix a map $f : X \to Y$. Thus:*

(1) *If f is continuous (resp. Borel measurable), then $\widehat{f} : \mathcal{N}(X) \to \mathcal{N}(Y)$ is continuous (resp. Borel measurable).*

(2) *If $f : X \to Y$ is a Borel measurable surjection, so is the induced map $\widehat{f} : \mathcal{N}(X) \to \mathcal{N}(Y)$. Additionally, if f is continuous and open, so is $\widehat{f}$.*

Proof: (1) Since $\widehat{f}$ is the combination of the functions $\left(\widehat{f}_{(n)}\right)_{n\in\mathbb{N}}$, where $\widehat{f}_{(n)} : \mathcal{N}_n(X) \to \mathcal{N}_n(Y)$, by Lemma 3, it is enough to show that, for each $n \in \mathbb{N}$, $\widehat{f}_{(n)}$ is continuous or Borel measurable. By [1, Theorem 15.14], the image measure map $\widetilde{f}$ is continuous, provided f is continuous. If f is assumed to be only Borel measurable, we conclude that $\widetilde{f}$ is Borel measurable by using two mathematical facts. First, the Borel σ-field on $\mathcal{M}(X)$ is generated by sets of the form $\{\mu \in \mathcal{M}(X) : \mu(E) \geq p\}$, where $E \in \Sigma_X$ and $p \in \mathbb{Q} \cap [0,1]$ (use [34, Theorem 17.24]). Second, each set $\widetilde{f}^{-1}(\{\nu \in \mathcal{M}(Y) : \nu(E) \geq p\})$ can be written as $\left\{\mu \in \mathcal{M}(X) : \mu\left(f^{-1}(E)\right) \geq p\right\}$. The conclusion that $\widehat{f}$ is continuous and/or Borel measurable follows from the fact that each space $\mathcal{N}_n(X)$ is endowed with the product topology.

(2) If $f : X \to Y$ is measurable and onto, then the map $\widetilde{f} : \mathcal{M}(X) \to \mathcal{M}(Y)$ is onto as a consequence of the Von Neumann Selection Theorem ([8, Theorem 91.15]), and this implies the desired conclusion. Furthermore, if f is continuous and open, then by [9, Corollary 2.1] it follows that the map $\widetilde{f} : \mathcal{M}(X) \to \mathcal{M}(Y)$ is a continuous, open surjection. An analogous conclusion holds for the map $\widehat{f} : \mathcal{N}(X) \to \mathcal{N}(Y)$ by virtue of Lemma 3. ■

Lemma 5 *Let X and Y be metrizable Lusin spaces, and fix a Borel measurable map $f : X \to Y$ and $\overline{\mu} = (\mu_1, ..., \mu_n) \in \mathcal{N}(X)$.*

(1) *If the image LPS $\widehat{f}(\overline{\mu})$ is mutually singular, so is $\overline{\mu}$.*

(2) *Let $\overline{\mu}$ be mutually singular. Suppose that the Borel sets $\{E_l\}_{l\leq n} \subseteq \Sigma_X$ satisfying the requirement of mutual singularity for $\overline{\mu}$ (Definition 1) are such that $E_l \in \sigma(f)$, for each $l \leq n$. Thus the image LPS $\widehat{f}(\overline{\mu})$ is mutually singular.*

(3) *Let $f : X \to Y$ be onto. Thus, for each $\overline{\nu} \in \mathcal{L}(Y)$, there exists $\overline{\mu} \in \mathcal{L}(X)$ such that $\widehat{f}(\overline{\mu}) = \overline{\nu}$.*

Proof: (1) If $\widehat{f}(\overline{\mu}) = \left(\widetilde{f}(\mu_1), ..., \widetilde{f}(\mu_n)\right)$ is mutually singular, then for each $l = 1, ..., n$, there are Borel sets E_l in Y such that $\mu_l\left(f^{-1}(E_l)\right) = 1$ and $\mu_l\left(f^{-1}(E_m)\right) = 0$ for $l \neq m$. Clearly, the collection $\left\{f^{-1}(E_l)\right\}_{l=1}^n \subseteq \Sigma_Y$ satisfies the required properties of mutual singularity for $\overline{\mu}$.

(2) By definition of $\sigma(f)$, for each E_l there exists $F_l \in \Sigma_Y$ such that $E_l = f^{-1}(F_l)$. The collection $\{F_l\}_{l=1}^n \subseteq \Sigma_Y$ satisfies the required properties of mutual singularity for $\widehat{f}(\overline{\mu})$.

(3) Let $\overline{\nu} \in \mathcal{L}(Y)$. By Lemma 4.(2), there exists $\overline{\mu} \in \mathcal{N}(X)$ such that $\widehat{f}(\overline{\mu}) = \overline{\nu}$. By part (1), $\overline{\mu} \in \mathcal{L}(X)$. ■

In what follows we shall make use of the following characterization of full-support LPS's.

Lemma 6 *Fix $\overline{\mu} = (\mu_1, ..., \mu_n) \in \mathcal{N}(X)$. The following are equivalent:*

(1) *$\overline{\mu}$ is of full-support.*

(2) *For each non-empty, open set $G \subseteq X$, there exists $l \in \mathbb{N}$, $l \leq n$, such that $\mu_l(G) > 0$.*

(3) *For each non-empty, basic open set $B \subseteq X$, there exists $l \in \mathbb{N}$, $l \leq n$, such that $\mu_l(B) > 0$.*

Proof: The equivalence (1)$\Longleftrightarrow$(2) is stated and proved in [15, Lemma C.1].
(2)$\Longrightarrow$(3). Obvious.
(3)$\Longrightarrow$(1). We prove the contrapositive. If $\overline{\mu}$ is not of full-support, then $U = X \diagdown (\cup_{l=1}^{n} \mathrm{Supp}\mu_l)$ is non-empty and open in X. Thus there exists a non-empty, open basic element B of X such that $B \subseteq U$. It turns out that $\mu_l(B) \leq \mu_l(U) = 0$ for each $l = 1, ..., n$. ■

Lemma 7 *Let X and Y be metrizable Lusin spaces, and fix a Borel measurable map $f : X \to Y$.*

(1) *If $\overline{\mu} \in \mathcal{N}(X)$ is of full-support and f is a continuous surjection, then $\widehat{f}(\overline{\mu})$ is of full-support.*

(2) *If X is finite (resp. countable), then for every $\overline{\mu} \in \mathcal{N}(X)$, the set $\mathrm{Supp}\widehat{f}(\overline{\mu}) \subseteq Y$ is of finite (resp. countable) cardinality.*

(3) *Let $\overline{\mu} \in \mathcal{N}(X)$. If $\widehat{f}(\overline{\mu})$ is of full-support and X is endowed with the coarsest topology such that f is continuous, then $\overline{\mu}$ is of full-support.*

Proof: (1) Suppose that $\overline{\mu}$ is of full-support, i.e., $X = \cup_{l=1}^{n} \mathrm{Supp}\mu_l$. For each $l = 1, ..., n$, it holds that

$$\mu_l\left(f^{-1}\left(\mathrm{Supp}\widetilde{f}(\mu_l)\right)\right) = 1,$$

so since the set $f^{-1}\left(\mathrm{Supp}\widetilde{f}(\mu_l)\right)$ is closed (by continuity of f)

$$\mathrm{Supp}\mu_l \subseteq f^{-1}\left(\mathrm{Supp}\widetilde{f}(\mu_l)\right).$$

It follows that

$$\begin{aligned} X &= \cup_{l=1}^{n} \mathrm{Supp}\mu_l \\ &\subseteq \cup_{l=1}^{n} f^{-1}\left(\mathrm{Supp}\widetilde{f}(\mu_l)\right) \\ &= f^{-1}\left(\cup_{l=1}^{n} \mathrm{Supp}\widetilde{f}(\mu_l)\right) \\ &\subseteq f^{-1}(Y) \\ &= X, \end{aligned}$$

hence

$$f^{-1}\left(\cup_{l=1}^{n} \mathrm{Supp}\widetilde{f}(\mu_l)\right) = f^{-1}(Y).$$

By the surjectivity of f we obtain

$$Y = \cup_{l=1}^{n} \mathrm{Supp}\widetilde{f}(\mu_l),$$

i.e., $\widehat{f}(\overline{\mu})$ is of full-support, as required.
(2) If X is finite (resp. countable), so is $f(X)$. Pick an arbitrary LPS $\overline{\mu} = (\mu_1, ..., \mu_n) \in \mathcal{N}(X)$. Then, for any $l = 1, ..., n$, the set $\mathrm{Supp}\mu_l$ has finite (resp. countable) cardinality, hence

$f(\mathrm{Supp}\mu_l)$ is a finite (resp. countable), closed subset of Y. Since $f^{-1}(f(\mathrm{Supp}\mu_l)) \supseteq \mathrm{Supp}\mu_l$, it holds that

$$\begin{aligned} \widetilde{f}(\mu_l)(f(\mathrm{Supp}\mu_l)) &= \mu_l\left(f^{-1}(f(\mathrm{Supp}\mu_l))\right) \\ &\geq \mu_l(\mathrm{Supp}\mu_l) \\ &= 1, \end{aligned}$$

thus $f(\mathrm{Supp}\mu_l) \supseteq \mathrm{Supp}\widetilde{f}(\mu_l)$, which implies that $\mathrm{Supp}\widetilde{f}(\mu_l)$ is a set of finite (resp. countable) cardinality, for each $l = 1, ..., n$. It follows that $\mathrm{Supp}\widehat{f}(\overline{\mu}) = \cup_{l=1}^{n}\mathrm{Supp}\widetilde{f}(\mu_l)$ has finite (resp. countable) cardinality.

(3) Let $\overline{\mu} = (\mu_1, ..., \mu_n) \in \mathcal{N}(X)$. Every open set $O \subseteq X$ is such that $O = f^{-1}(U)$ for some open set $U \subseteq X$. Since $\widehat{f}(\overline{\mu})$ is of full-support, then there exists $l \leq n$ such that

$$\begin{aligned} \mu_l(O) &= \mu_l\left(f^{-1}(U)\right) \\ &= \widetilde{f}(\mu_l)(O) \\ &> 0, \end{aligned}$$

and this shows that $\overline{\mu}$ is of full-support. ■

5.1.2 Structure of the spaces of LPS's

Recall that a set U in a topological space X is a G_δ**-set** if it is a countable intersection of open subsets of X. It is easy to check that the family of G_δ-sets in a topological space is closed under countable intersections and *finite* unions. A set F is an F_σ**-set** if its complement $X \diagdown F$ is a G_δ-set. A set $G \subseteq X$ is **ambivalent** if it is both a G_δ-set and F_σ-set in X (see, e.g., [43]). If X is a metrizable topological space, then both closed and open subsets of X are ambivalent.

Lemma 8 *Fix a topological space X.*

(i) *If X is metrizable Lusin (resp. Polish), then $\mathcal{N}(X)$ is metrizable Lusin (resp. Polish).*

(ii) *If X is metrizable Lusin (resp. Polish), then $\mathcal{L}(X)$ is a G_δ-subset (so Borel) of $\mathcal{N}(X)$, so metrizable Lusin (resp. Polish) in the relative topology.*

To prove Lemma 8, we need the following result on mutually singular probability measures:

Claim 1 *Let X be a metrizable Lusin space. Two Borel probability measures $\mu, \nu \in \mathcal{M}(X)$ are mutually singular if and only if for each $k \in \mathbb{N}$, there exists a compact set $K \subseteq X$ such that $\mu(K) < \frac{1}{2^k}$ and $\nu(K) > 1 - \frac{1}{2^k}$.*[23]

Proof: (Necessity) Let $B \in \Sigma_X$ such that $\mu(B) = 0$ and $\nu(B) = 1$. Every Borel probability measure on a Lusin space is Radon ([49, Theorem 10, pp.122-124]), so for each $k \in \mathbb{N}$, there exists a compact set $K \subseteq B$ such that $\mu(K) = 0$ and $\nu(B \diagdown K) < \frac{1}{2^k}$, which implies $\nu(K) = \nu(B) - \nu(B \diagdown K) > 1 - \frac{1}{2^k}$.

[23] Note that this result is *not* true if the requirement that X be Lusin is dropped. In such a case, a weaker result is true, namely, that the set K in the statement of Lemma 1 is simply Borel (see [32, Footnote 2] and [6, Footnote 2]).

(Sufficiency) For each $n \in \mathbb{N}$, let K_n be a compact set such that $\mu(K_n) < \frac{1}{2^n}$ and $\nu(K_n) > 1 - \frac{1}{2^n}$. For each $n \in \mathbb{N}$, $\nu(\cup_{m \geq n} K_n) = 1$. Thus, the set $B = \cap_{n \in \mathbb{N}} \cup_{m \geq n} K_n$ is non-empty and Borel and satisfies $\mu(B) = 0$ and $\nu(B) = 1$. ■

We also make use of the following properties of subsets of $\mathcal{M}(X)$ on a separable and metrizable space X.

Claim 2 *Let K be a compact subset of a metrizable Lusin space X. Thus, for each $p \in \mathbb{Q} \cap [0,1]$, sets of the form*

$$\{\mu \in \mathcal{M}(X) \,|\mu(K) < p\},$$
$$\{\mu \in \mathcal{M}(X) \,|\mu(K) > p\},$$

are open and ambivalent subsets of $\mathcal{M}(X)$, respectively.

To prove Claim 2, we recall that a real-valued function f on a metrizable space X is **upper** (resp. **lower**) **semicontinuous** if the set $f^{-1}([c, +\infty))$ (resp. $f^{-1}((-\infty, c])$) is closed in X for every $c \in \mathbb{R}$. A function $f : X \to \mathbb{R}$ is a **Baire Class 1** function if $f^{-1}(O)$ is an F_σ-set (resp. G_δ-set) in X provided O is open (resp. closed) in $\mathbb{R}$. A semicontinuous function is also a Baire Class 1 function.

Proof of Claim 2: The *weak**-topology on $\mathcal{M}(X)$ is the coarsest topology such that each function $\mu \longmapsto \int f d\mu$ is lower (resp. upper) semicontinuous whenever $f : X \to \mathbb{R}$ is lower (resp. upper) semicontinuous (cf. [52, Theorem 8.1] or [49, Appendix]; see also [1, Theorem 15.5]). Since indicator functions on open (resp. closed) sets are lower (resp. upper) semicontinuous functions, it follows that the evaluation map $e_A : \mathcal{M}(X) \to [0,1]$ defined as

$$e_A(\mu) = \int \mathbf{1}_A d\mu = \mu(A), \; \mu \in \mathcal{M}(X), \; A \in \Sigma_X,$$

is lower (resp. upper) semicontinuous if A is open (resp. closed) in X. Fix $p \in \mathbb{Q} \cap [0,1]$ and a compact (so closed) set $K \subseteq X$. The set $\{\mu \in \mathcal{M}(X) \,|\mu(K) < p\}$ can be written as

$$\{\mu \in \mathcal{M}(X) \,|\mu(K) < p \;\} = e_K^{-1}([0,p)),$$

i.e., $e_K^{-1}([0,p))$ is the inverse image of the set $[0,p)$, open in $[0,1]$, under an upper semicontinuous map, hence it is open in $\mathcal{M}(X)$. Note that

$$\{\mu \in \mathcal{M}(X) \,|\mu(K) > p \;\} = \cup_{k=1}^{\infty} \left\{ \mu \in \mathcal{M}(X) \middle| \mu(K) \geq p + \frac{1}{k} \;\right\},$$

so $\{\mu \in \mathcal{M}(X) \,|\mu(K) > p \;\}$ is a countable union of closed sets, hence an F_σ-set. We show that it is also a G_δ-set. To this end, note that we can write

$$\begin{aligned} \{\mu \in \mathcal{M}(X) \,|\mu(K) > p \;\} &= \{\mu \in \mathcal{M}(X) \,|\mu(X \setminus K) \leq 1 - p \;\} \\ &= e_{X \setminus K}^{-1}([0, 1-p]), \end{aligned}$$

and since $X \setminus K$ is open in X, the map $e_{X \setminus K} : \mathcal{M}(X) \to [0,1]$ is lower semicontinuous. In particular, the map $e_{X \setminus K}$ is of Baire Class 1, hence $e_{X \setminus K}^{-1}([0, 1-p])$ is a G_δ-subset of $\mathcal{M}(X)$ in that the set $[0, 1-p]$ is closed in $[0,1]$. This shows that $\{\mu \in \mathcal{M}(X) \,|\mu(K) > p \;\}$ is an ambivalent set, as required. ■

Finally, we recall that, given a countable collection of pairwise disjoint topological spaces $\{X_n\}_{n\in\mathbb{N}}$, a set A is open (resp. closed) in $X = \cup_{n\in\mathbb{N}} X_n$ if and only if, for all $n \in \mathbb{N}$, $A \cap X_n$ is an open (resp. closed) subset of X_n. The following Claim states an analogous result concerning ambivalent sets.

Claim 3 *Let $\{X_n\}_{n\in\mathbb{N}}$ be a countable collection of pairwise disjoint topological spaces. The set G is a G_δ-subset (resp. F_σ-subset) of $X = \cup_{n\in\mathbb{N}} X_n$ if and only if, for all $n \in \mathbb{N}$, $G \cap X_n$ is a G_δ-subset (resp. F_σ-subset) of X_n.*[24]

Proof: We prove the statement for the case in which G is a G_δ-set. When G is a F_σ-set, the statement follows simply from the fact that a set is F_σ if and only if the complement is G_δ.

(Necessity) Let $G = \cap_{k\in\mathbb{N}} O_k$ with each O_k open in X. So, for all $n \in \mathbb{N}$, $O_k \cap X_n$ is an open subset of X_n. It follows that, for all $n \in \mathbb{N}$,

$$\begin{aligned} G \cap X_n &= (\cap_{k\in\mathbb{N}} O_k) \cap X_n \\ &= \cap_{k\in\mathbb{N}} (O_k \cap X_n), \end{aligned}$$

i.e., $G \cap X_n$ is G_δ in X_n.

(Sufficiency) If $G \cap X_n$ is G_δ in X_n, then $G \cap X_n = \cap_{l\in\mathbb{N}} O_l^n$, where each O_l^n is open in X_n. As such, the set $O_l = \cup_{n\in\mathbb{N}} O_l^n$ is open in X, for all $l \in \mathbb{N}$. The set G can be written as countable intersection of open subsets of X as follows:

$$\begin{aligned} G &= \cup_{n\in\mathbb{N}} (\cap_{l\in\mathbb{N}} O_l^n) \\ &= \cap_{l\in\mathbb{N}} (\cup_{n\in\mathbb{N}} O_l^n) \\ &= \cap_{l\in\mathbb{N}} O_l, \end{aligned}$$

where the second equality comes from disjointness of $\{X_n\}_{n\in\mathbb{N}}$. ■

Proof of Lemma 8: Part (i): If X is Lusin (resp. Polish), then $\mathcal{M}(X)$ is Lusin (resp. Polish). Consequently, the product topology on each $(\mathcal{M}(X))^n$ is Lusin (resp. Polish), so the topological sum $\cup_{n\in\mathbb{N}} (\mathcal{M}(X))^n$ is Lusin (resp. Polish).

Part (ii): For $l, m \in \mathbb{N}$, $l \neq m$, let

$$\mathcal{L}_n^{l,m}(X) = \{(\mu_1, ..., \mu_n) \in \mathcal{N}_n(X) \,|\mu_l \perp \mu_m\}.$$

(The symbol "$\perp$" denotes the mutual singularity of probability measures.) Thus $\mathcal{L}_n(X) = \cap_{m=1}^n \cap_{l\neq m} \mathcal{L}_n^{l,m}(X)$, so that $\mathcal{L}(X) = \cup_{n\in\mathbb{N}} \mathcal{L}_n(X)$. We show that each $\mathcal{L}_n^{l,m}(X)$ is a G_δ-subset (so Borel) of $\mathcal{N}(X)$. Using Claim 3, it will follow that $\mathcal{L}(X)$ is a G_δ-set in $\mathcal{N}(X)$, as required. By Claim 1 we can write $\mathcal{L}_n^{l,m}(X)$ as

$$\mathcal{L}_n^{l,m}(X) = \left\{ (\mu_1, ..., \mu_n) \in \mathcal{N}_n(X) \,\middle|\, \forall k \in \mathbb{N}, \mu_l(K_k) < \frac{1}{2^k}, \mu_m(K_k) > 1 - \frac{1}{2^k} \right\},$$

where $\{K_k\}_{k\in\mathbb{N}}$ is a collection of compact subsets of X. If X is Lusin, so is $\mathcal{M}(X)$, and sets of the form

$$\left\{ \mu \in \mathcal{M}(X) \,\middle|\, \mu(K_k) < \frac{1}{2^k} \right\},$$
$$\left\{ \mu \in \mathcal{M}(X) \,\middle|\, \mu(K_k) > 1 - \frac{1}{2^k} \right\},$$

[24] The result in Claim 3 should be known, but we did not find any reference about them, so a (simple) proof is provided.

are, respectively, open and G_δ in $\mathcal{M}(X)$ by Claim 2. By continuity of projection maps $(\mu_1, ..., \mu_n) \mapsto \mu_l$, the sets

$$V_l(K_k) = \left\{ (\mu_1, ..., \mu_n) \in \mathcal{N}_n(X) \left| \mu_l(K_k) < \frac{1}{2^k} \right. \right\},$$
$$V_m(K_k) = \left\{ (\mu_1, ..., \mu_n) \in \mathcal{N}_n(X) \left| \mu_m(K_k) > 1 - \frac{1}{2^k} \right. \right\},$$

are ambivalent subsets of $\mathcal{N}_n(X)$ - specifically, $V_l(K_k)$ is an open cylinder (hence G_δ), while $V_m(K_k)$ is a cylinder with a G_δ base which is both a G_δ-set and an F_σ-set, so an ambivalent set (see [25, Exercise 2.3.B.(b)]). It follows that $\mathcal{L}_n^{l,m}(X) = \cap_{k\in\mathbb{N}}(V_l(K_k) \cap V_m(K_k))$ is a countable intersection of G_δ-sets, hence a G_δ-subset of $\mathcal{N}(X)$.

Finally, if X is Polish, part (i) gives that $\mathcal{N}(X)$ is also Polish. The conclusion that $\mathcal{L}(X)$ is Polish in the relative topology follows from the fact that $\mathcal{L}(X)$ is a G_δ-subset of $\mathcal{N}(X)$. ■

Two remarks on the results stated in Lemma 8 are in order. First, Burgess and Mauldin ([16, Theorem 2]) show that if X is a compact metrizable space, then $\mathcal{L}_2(X)$ is a G_δ-subset of $\mathcal{N}_2(X)$.[25] As the Authors point out ([16, p.904]), such result remains true if X is only assumed to be Polish. Thus Lemma 8,(ii) provides a generalization of the result in [16] with a proof which is, in our view, simpler than the original one.

Second, the result in Lemma 8 concerning the topological structure of $\mathcal{L}(X)$ appears to be tight. We note that, in general, the set $\mathcal{L}(X)$ is neither closed nor open in $\mathcal{N}(X)$, as the following example shows.

Example 3 *Let $X = \mathbb{R}$, and consider a sequence of LPS's $\{\overline{\mu}_n = (\nu_n, \lambda_n)\}_{n\in\mathbb{N}}$ where $\nu_n = \delta_0$ (i.e., Dirac point mass at 0) for all $n \in \mathbb{N}$, and each λ_n is described by a uniform pdf on $\left[-\frac{1}{n}, \frac{1}{n}\right]$. Clearly, each $\overline{\mu}_n \in \mathcal{L}(X)$, but $\overline{\mu}_n \longrightarrow (\delta_0, \delta_0) \notin \mathcal{L}(X)$. So $\mathcal{L}(X)$ is not closed in $\mathcal{N}(X)$.*

To see that $\mathcal{L}(X)$ is not open in $\mathcal{N}(X)$, we show that $\mathcal{N}(X) \diagdown \mathcal{L}(X)$ is not closed. As before, let $X = \mathbb{R}$, and consider the sequence of LPS's $\{\overline{\mu}_n = (\nu_n, \lambda_n)\}_{n\in\mathbb{N}}$ where, for all $n \in \mathbb{N}$, $\lambda_n = \lambda$ is the Lebesgue measure on $[0, 1]$, and each ν_n is a Gaussian measure with mean 0 and standard deviation $\frac{1}{n}$. Each ν_n is absolutely continuous with respect to λ, so $\overline{\mu}_n \notin \mathcal{L}(X)$ for all $n \in \mathbb{N}$, but $\overline{\mu}_n \longrightarrow (\delta_0, \lambda) \in \mathcal{L}(X)$.

However, $\mathcal{L}(X)$ turns out to be closed in $\mathcal{N}(X)$ provided X is countable.

Corollary 2 *If X is a countable Lusin space, then $\mathcal{L}(X)$ is a closed subset of $\mathcal{N}(X)$.*

Proof: If X is countable (so Polish), then, for all $A \subseteq X$, the evaluation map $e_A : \mathcal{M}(X) \to [0, 1]$ defined as $e_A(\mu) = \mu(A)$ is continuous. As such, for each $p \in \mathbb{Q} \cap [0, 1]$, sets of the form

$$e_A^{-1}(\{p\}) = \{\mu \in \mathcal{M}(X) \,|\mu(A) = p \}, \, A \subseteq X,$$

are closed. Proceeding as in the proof of Lemma 8.(ii), it is easily seen that the set

$$\mathcal{L}_n^{l,m}(X) = \bigcap_{A\subseteq X} \{(\mu_1, ..., \mu_n) \in \mathcal{N}_n(X) \,|\mu_l(A) = 0, \mu_m(A) = 1\}$$

[25] In fact, Theorem 2 in [16] shows that $\mathcal{L}_2(X)$ is a G_δ-subset of $\mathcal{N}_2(X) \diagdown \Delta_2(X)$, where $\Delta_2(X)$ stands for the "diagonal" of $\mathcal{N}_2(X)$, formally $\Delta_2(X) = \{(\mu_1, \mu_2) \in \mathcal{N}_2(X) \,|\mu_1 = \mu_2 \}$. It is straighforward to check that $\Delta_2(X)$ is closed in $\mathcal{N}_2(X)$.

is closed, which in turn implies that $\mathcal{L}_n(X) = \cap_{m=1}^{n} \cap_{l \neq m} \mathcal{L}_n^{l,m}(X)$ is also closed in $\mathcal{N}(X)$, for all $n \in \mathbb{N}$. It turns out $\mathcal{L}(X) = \cup_{n \in \mathbb{N}} \mathcal{L}_n(X)$ is a closed subset of $\mathcal{N}(X)$, by the property of the direct sum topology listed above. ■

Lemma 9 *Fix a metrizable Lusin space X. If $F \subseteq X$ is non-empty and Borel, then $\mathcal{N}(F)$ is homeomorphic to $\{(\mu_1, ..., \mu_n) \in \mathcal{N}(X) \,|\mu_l(F) = 1, \forall l \leq n\}$. Analogously, the space $\mathcal{L}(F)$ is homeomorphic to $\{(\mu_1, ..., \mu_n) \in \mathcal{L}(X) \,|\mu_l(F) = 1, \forall l \leq n\}$.*

Proof: If F is a non-empty Borel subset of a metrizable space X, then $\mathcal{M}(F)$ is homeomorphic to $\{\mu \in \mathcal{M}(X) \,|\mu(F) = 1\}$ ([34, p.114, Exercise 17.28]). So, for $n \in \mathbb{N}$, it turns out that the set $\mathcal{N}_n(F) = (\mathcal{M}(X))^n$ is homeomorphic to $\mathcal{F}_n = \{\overline{\mu} \in \mathcal{N}_n(X) \,|\mu_l(F) = 1, \forall l \leq n\}$. By definition of topological sum, it turns out that $\mathcal{N}(F) = \cup_{n \in \mathbb{N}} \mathcal{N}_n(F)$ is homeomorphic to $\cup_{n \in \mathbb{N}} \mathcal{F}_n$. By this, it follows that $\mathcal{L}(X) \cap (\cup_{n \in \mathbb{N}} \mathcal{F}_n)$ is homeomorphic to $\mathcal{L}(X) \cap \mathcal{N}(F) = \mathcal{L}(F)$. ■

We finally list some properties of the Borel σ-field of the spaces $\mathcal{N}(X)$ and $\mathcal{L}(X)$.

Given a measurable space $(X, \mathcal{A}_X)$, where X is not necessarily given a topological structure (hence $\mathcal{A}_X$ is an arbitrary σ-field), let $\mathcal{A}_{\mathcal{M}(X)}$ denote the σ-field on $\mathcal{M}(X)$ generated by all sets of the form

$$b^p(E) = \{\mu \in \mathcal{M}(X) : \mu(E) \geq p\}$$

where $E \in \mathcal{A}$ and $p \in \mathbb{Q} \cap [0, 1]$. Alternatively put, the σ-field $\mathcal{A}_{\mathcal{M}(X)}$ is the restriction to $\mathcal{M}(X)$ of the σ-field generated by the Borel cylinders in $[0, 1]^{\Sigma_X}$ (i.e., the σ-field generated by maps $\mu \mapsto \mu(E)$, for all $E \in \Sigma_X$).

Given a countable family of pairwise disjoint measurable spaces $\{(X_n, \mathcal{A}_{X_n})\}_{n \in \mathbb{N}}$, let $X = \cup_{n \in \mathbb{N}} X_n$. Write $e_n : X_n \to X$ for the canonical injection. For a set $E \subseteq X$, $e_n^{-1}(E) = E \cap X_n$. Thus, the **direct sum** of the measurable spaces $\{(X_n, \mathcal{A}_{X_n})\}_{n \in \mathbb{N}}$ is defined as the the finest σ-field $\mathcal{A}_X$ on X for which each canonical injection is measurable (that is, $\mathcal{A}_X$ is the final σ-field on X for the family of mappings e_n), formally:

$$\begin{aligned} \mathcal{A}_X &= \left\{E \subseteq X \left| e_n^{-1}(E) \in \mathcal{A}_{X_n}, \forall n \in \mathbb{N}\right.\right\} \\ &= \cap_{n \in \mathbb{N}} \{E \subseteq X \,|E \cap X_n \in \mathcal{A}_{X_n}\}. \end{aligned}$$

Evidently $\mathcal{A}_{X_n} = \{E \in \mathcal{A}_X \,|E \subseteq X_n\}$, and $(X_n, \mathcal{A}_{X_n})$ is a measurable subspace of $(X, \mathcal{A}_X)$. So a set $E \subseteq X$ belongs to $\mathcal{A}_X$ if and only if it can be written as $E = \cup_{n \in \mathbb{N}} E_n$, where $E_n = E \cap X_n \in \mathcal{A}_{X_n}$ for all $n \in \mathbb{N}$. Note that, if X is endowed with the direct sum σ-field $\mathcal{A}_X$, then each canonical injection $e_m : X_m \to X$ is a measure-theoretic embedding.

The following result is easy to prove:

Lemma 10 *Let $\{X_n\}_{n \in \mathbb{N}}$ be a countable family of topological spaces, and let $X = \cup_{n \in \mathbb{N}} X_n$ be endowed with the topological sum. For all $n \in \mathbb{N}$, let Σ_{X_n} be the Borel σ-field of the space X_n. Then the direct sum σ-field $\mathcal{A}_X$ of the measurable spaces $\{(X_n, \Sigma_{X_n})\}_{n \in \mathbb{N}}$ equals the Borel σ-field on X generated by the topological sum.*

We also provide a result for generators of the direct sum σ-field.

Lemma 11 [26] *Let $\{(X_n, \mathcal{A}_{X_n})\}_{n\in\mathbb{N}}$ be a countable family of pairwise disjoint measurable spaces. Suppose that, for all $n \in \mathbb{N}$, $\mathcal{F}_{X_n}$ is a field of subsets of X_n generating $\mathcal{A}_{X_n}$. Thus*

$$\mathcal{A}_X = \sigma\left(\cup_{n\in\mathbb{N}}\mathcal{F}_{X_n}\right).$$

If $\mathcal{G}$ is a family of subsets of X and $E \subseteq X$, we write $\mathcal{G} \cap E = \{F \cap E \,|F \in \mathcal{G}\}$. We write $\sigma(\mathcal{G} \cap E)$ for the σ-field of subsets of E generated by the family $\mathcal{G} \cap E$ of subsets of E. The proof of Lemma 11 makes use of the following well-known result ([54, Theorem 1.15]), namely

$$\sigma(\mathcal{G} \cap E) = \sigma(\mathcal{G}) \cap E. \tag{5.1}$$

Proof: The set containment $\sigma\left(\cup_{n\in\mathbb{N}}\mathcal{F}_{X_n}\right) \subseteq \mathcal{A}_X$ is obvious, in view of the fact that $\mathcal{F}_{X_n} \subseteq \mathcal{A}_{X_n}$ for all $n \in \mathbb{N}$. To show that $\mathcal{A}_X \subseteq \sigma\left(\cup_{n\in\mathbb{N}}\mathcal{F}_{X_n}\right)$, pick any $F \in \mathcal{A}_X$. Thus, by definition of direct sum σ-field, $F = \cup_{n\in\mathbb{N}}F_n$ where $F_n = F \cap X_n \in \sigma(\mathcal{F}_{X_n}) = \mathcal{A}_{X_n}$ for all $n \in \mathbb{N}$. It is immediate to check that

$$\mathcal{F}_{X_m} = \left(\cup_{n\in\mathbb{N}}\mathcal{F}_{X_n}\right) \cap X_m, \forall m \in \mathbb{N}.$$

It follows from (5.1) that

$$\begin{aligned} \mathcal{A}_{X_m} &= \sigma(\mathcal{F}_{X_m}) \\ &= \sigma\left(\left(\cup_{n\in\mathbb{N}}\mathcal{F}_{X_n}\right) \cap X_m\right) \\ &= \sigma\left(\cup_{n\in\mathbb{N}}\mathcal{F}_{X_n}\right) \cap X_m, \end{aligned}$$

for all $m \in \mathbb{N}$. Thus, if $F_m \in \mathcal{A}_{X_m}$, then $F_m = E_m \cap X_m$ for some $E_m \in \sigma\left(\cup_{n\in\mathbb{N}}\mathcal{F}_{X_n}\right)$. Since each $\mathcal{F}_{X_n}$ is a field, then $X_m \in \mathcal{F}_{X_m}$, so $X_m \in \cup_{n\in\mathbb{N}}\mathcal{F}_{X_n} \subseteq \sigma\left(\cup_{n\in\mathbb{N}}\mathcal{F}_{X_n}\right)$. This in turn implies that $F_m \in \sigma\left(\cup_{n\in\mathbb{N}}\mathcal{F}_{X_n}\right)$ for all $m \in \mathbb{N}$. Hence $F = \cup_{n\in\mathbb{N}}F_n \in \sigma\left(\cup_{n\in\mathbb{N}}\mathcal{F}_{X_n}\right)$, and this concludes the proof. ■

The measurable space $\left(\mathcal{N}_n(X), \mathcal{A}_{\mathcal{N}_n(X)}\right)$ of length-n LPS's on $(X, \mathcal{A}_X)$ is defined as follows: $\mathcal{N}_n(X) = (\mathcal{M}(X))^n$ and $\mathcal{A}_{\mathcal{N}_n(X)}$ is the product σ-field. So the space $\left(\mathcal{N}(X), \mathcal{A}_{\mathcal{N}(X)}\right)$ of length-n LPS's on $(X, \mathcal{A}_X)$ is such that $\mathcal{N}(X) = \cup_{n\in\mathbb{N}}\mathcal{N}_n(X)$ and $\mathcal{A}_{\mathcal{N}(X)}$ is the direct sum σ-field.

Lemma 12 *Fix a measurable space $(X, \mathcal{A}_X)$. The σ-field $\mathcal{A}_{\mathcal{N}(X)}$ on $\mathcal{N}(X)$ is generated by sets of the form*

$$\{(\mu_1, ..., \mu_n) \in \mathcal{N}(X) \,|\mu_l(A) \geq p_l, \forall l \leq n\},$$

where $A \in \mathcal{A}_X$ and $p_l \in \mathbb{Q} \cap [0,1]$ for all $l \leq n$. Additionally, if X is a separable and metrizable space, $\mathcal{A}_X$ is its Borel σ-field and the space $\mathcal{M}(X)$ is endowed with the weak-topology, then $\mathcal{A}_{\mathcal{N}(X)}$ equals the Borel σ-field $\Sigma_{\mathcal{N}(X)}$ of the topological space $\mathcal{N}(X)$.*

Proof: The σ-field $\mathcal{A}_{\mathcal{M}(X)}$ is generated by sets of the form $\{\mu \in \mathcal{M}(X) : \mu(A) \geq p\}$, where $A \in \mathcal{A}_X$ and $p \in \mathbb{Q} \cap [0,1]$. So, for each $n \in \mathbb{N}$, sets of the form

$$\{(\mu_1, ..., \mu_n) \in \mathcal{N}_n(X) \,|\mu_l(A) \geq p_l, \forall l \leq n\}$$

[26] We did not find any reference to this result, which should be known. We point out that a similar result can be found in [36, Proposition 2.8] with different (i.e., weaker) assumptions concerning the generators of the σ-fields. However, the result in [36] is stated and proved with just two factor spaces.

where $A \in \mathcal{A}_X$ and $p_l \in \mathbb{Q} \cap [0,1]$ for all $l \leq n$, generate the product σ-field $\mathcal{A}_{\mathcal{N}_n(X)}$. Let $\mathcal{F}_{\mathcal{N}_n(X)}$ denote the collection of such sets. By Lemma 11, $\mathcal{A}_{\mathcal{N}(X)}$ is generated by the family $\cup_{n\in\mathbb{N}}\mathcal{F}_{\mathcal{N}_n(X)}$. A set belonging to $\cup_{n\in\mathbb{N}}\mathcal{F}_{\mathcal{N}_n(X)}$ can be written as

$$\{(\mu_1, ..., \mu_n) \in \mathcal{N}(X) \,|\, \mu_l(A) \geq p_l, \forall l \leq n\}.$$

where $A \in \mathcal{A}_X$ and $p_l \in \mathbb{Q} \cap [0,1]$ for all $l \leq n$.

If X is a separable and metrizable space, so is the space $\mathcal{M}(X)$ endowed with the weak* topology ([1, Theorem 15.12]). As such, the Borel σ-field $\Sigma_{\mathcal{M}(X)}$ on $\mathcal{M}(X)$ generated by the *weak**-topology equals $\mathcal{A}_{\mathcal{M}(X)}$ by [34, Theorem 17.24]. Since $\mathcal{M}(X)$ is also second countable, then, for all $n \in \mathbb{N}$, the Borel σ-field $\Sigma_{\mathcal{N}_n(X)}$ generated by the product topology on $\mathcal{N}_n(X) = (\mathcal{M}(X))^n$ coincides with the product of the σ-fields $\Sigma_{\mathcal{M}(X)}$ ([1, Theorem 4.44]). Hence $\Sigma_{\mathcal{N}_n(X)} = \mathcal{A}_{\mathcal{N}_n(X)}$, and the conclusion $\Sigma_{\mathcal{N}(X)} = \mathcal{A}_{\mathcal{N}(X)}$ follows from Lemma 10. ■

Given a measurable space $(X, \mathcal{A}_X)$, let $\mathcal{F}_X$ be a non-empty system of generators of $\mathcal{A}_X$. Heifetz and Samet ([29, Lemma 4.5]) show that, if $\mathcal{F}_X$ is a field, then $\mathcal{A}_{\mathcal{M}(X)}$ is generated by sets of the form

$$b^p(E) = \{\mu \in \mathcal{M}(X) : \mu(E) \geq p\}$$

where $A \in \mathcal{F}_X$ and $p \in \mathbb{Q} \cap [0,1]$.[27] As such, the following result is immediate.

Corollary 3 *Given a measurable space $(X, \mathcal{A}_X)$, let $\mathcal{F}_X$ be a field of subsets of X generating the σ-field $\mathcal{A}_X$. Thus $\mathcal{A}_{\mathcal{N}(X)}$ is generated by sets of the form*

$$\{(\mu_1, ..., \mu_n) \in \mathcal{N}(X) \,|\, \mu_l(A) \geq p_l, \forall l \leq n\},$$

where $A \in \mathcal{F}_X$ and $p_l \in \mathbb{Q} \cap [0,1]$ for all $l \leq n$.

5.1.3 Projective systems of LPS's

We provide here some terminology and results for the theory of projective limits, especially as they relate to LPS's, and prove results which are needed in the proof of Lemma 2 in Section 3. For a more thorough treatment see [25] or [46].

Definition 17 *Let $\{X_p\}_{p\geq 1}$ be a countable family of metrizable Lusin spaces, and for each p let $\overline{\mu}^p = (\mu_1^p, ..., \mu_n^p)$ be a LPS over (the Borel σ-field Σ_{X_p} of) X_p. Suppose that, for each $p \leq q$, there exists a continuous function $\pi_{p,q} : X_q \to X_p$ such that*

(i) *$\pi_{p,r} = \pi_{p,q} \circ \pi_{q,r}$ whenever $p \leq q \leq r$, and $\pi_{p,p}$ is the identity;*

(ii) *$\pi_{p,q}$ is continuous and onto;*

(iii) *$\widehat{\pi}_{p,q}(\overline{\mu}^q) = \overline{\mu}^p$, i.e., $\widetilde{\pi}_{p,q}(\mu_l^q) = \mu_l^p$ for all $l = 1, ..., n$.*

Then we say that the collection $\mathcal{P} = (X_p, \pi_{p,q})_{p\geq 1,\ q\geq p}$ is a projective system of metrizable Lusin spaces, and $\mathcal{P}_{LPS} = (X_p, \pi_{p,q}, \overline{\mu}^p)_{p\geq 1,\ q\geq p}$ is a projective system of LPS's. If each $\overline{\mu}^q$ is a length-1 LPS, then we call $\mathcal{P}_{LPS}$ a projective system of probability measures.

[27] Actually, [29, Lemma 4.5] states the result for $p \in [0,1]$. This difference is, however, immaterial.

Definition 18 *Fix a projective system of metrizable Lusin spaces $\mathcal{P} = (X_p, \pi_{p,q})_{p\geq 1,\ q\geq p}$. The set*

$$X = \left\{ (x_p)_{p\geq 1} \in \prod_{p=1}^{\infty} X_p \,|\pi_{p,q}(x_q) = x_p, \forall q \geq p \right\}$$

is called the projective limit set of $\mathcal{P}$. The map $\pi_q : X \to X_q$ given by $\pi_q(x) = x_q$, $q \geq 1$, is called canonical projection, and is the restriction of the projection map $\mathrm{Pr}_{X_q} : \Pi_{p\geq 1} X_p \to X_q$ to X. Thus $(X, \pi_p)_{p\geq 1}$ is called the projective limit of $\mathcal{P}$.

The following result is standard.

Proposition 11 *Let $\mathcal{P} = (X_p, \pi_{p,q})_{p\geq 1,\ q\geq p}$ be a projective system of metrizable Lusin spaces. Then the projective limit $(X, \pi_p)_{p\geq 1}$ of $\mathcal{P}$ exists (i.e., X is non-empty). The projective limit set X is a metrizable Lusin space, and the collection of all subsets of X of the form $\pi_p^{-1}(O_p)$ with O_p open in X_p is a basis for the topology of X.*

Proof: Since each function $\pi_{p,q} : X_q \to X_p$ is onto, it follows from [12, Proposition 5], that the projective limit set X is non-empty. Moreover, X is a closed subset of the product topological space $\Pi_{p\geq 1} X_p$, so X is metrizable Lusin in the relative topology. For the last statement of the Proposition, apply [31, Theorem 158]. ■

Corollary 4 *Let $\mathcal{P} = (X_p, \pi_{p,q})_{p\geq 1,\ q\geq p}$ be a projective system of metrizable Lusin spaces. Thus the Borel σ-field of projective limit set X is $\Sigma_X = \sigma(\mathcal{F}_X)$, where $\mathcal{F}_X = \cup_{p\geq 1} \pi_p^{-1}(\Sigma_{X_p})$ is the field generated by the measurable cylinders (i.e., $\mathcal{F}_X$ is the cylindrical field).*

Next:

Definition 19 *Let $\mathcal{P}_{LPS} = (X_p, \pi_{p,q}, \overline{\mu}^p)_{p\geq 1,\ q\geq p}$ be a projective system of LPS's. Say $(X, \pi_p, \overline{\mu})_{p\geq 1}$ is the projective limit of $\mathcal{P}_{LPS}$ if*

(i) *$(X, \pi_p)_{p\geq 1}$ is the projective limit of the projective system $\mathcal{P} = (X_p, \pi_{p,q})_{p\geq 1,\ q\geq p}$.*

(ii) *$\overline{\mu}$ is a LPS (called limit LPS) defined on (X, Σ_X) such that*

$$\widehat{\pi}_p(\overline{\mu}) = \overline{\mu}^p,$$

for each $p \geq 1$.

Having defined the notion of projective limit of projective sequences of LPS's, we can state and prove the main result.

Theorem 4 *Let $\mathcal{P}_{LPS} = (X_p, \pi_{p,q}, \overline{\mu}^p)_{p\geq 1,\ q\geq p}$ be a projective system of LPS's. Thus, the projective limit $(X, \pi_p, \overline{\mu})_{p\geq 1}$ of $\mathcal{P}_{LPS}$ exists and is unique. Furthermore*

(i) *If there exists $p^* \geq 1$ such that $\overline{\mu}^{p^*}$ is mutually singular, then $\overline{\mu}$ is mutually singular.*

(ii) *$\overline{\mu}$ is of full-support if and only if $\overline{\mu}^p$ is of full-support, for each $p \geq 1$.*

Finally, we mention the following generalized version of Kolmogorov Existence Theorem, whose proof can be found in [11, pp.53-54] or [49, Theorem 21 and Corollary]. Recall that a Borel probability measure μ on a topological space X is *Radon* if for every Borel set A and every $\epsilon > 0$, there exists a compact set $K \subseteq A$ such that $\mu(A \backslash K) < \epsilon$.

Theorem 5 *Let $\mathcal{P} = (X_p, \pi_{p,q}, \mu^q)_{p\geq 1,\ q\geq p}$ be a projective system of probability measures such that each X_p is a Hausdorff topological space and each μ^q is Radon. Then the projective limit $(X, \pi_p, \mu)_{p\geq 1}$ exists and μ is a unique Radon probability measure.*

Proof of Theorem 4: Every Borel probability measure on a Lusin space is Radon ([49, Theorem 10, pp.122-124]), so by Kolmogorov Existence Theorem (Theorem 5) it follows that there exists a unique LPS $\overline{\mu} = (\mu_1, ..., \mu_n)$ where each μ_l is a probability measure on (X, Σ_X) such that

$$\widetilde{\pi}_p(\mu_l) = \mu_l^p,$$

for each $p \geq 1$. By Corollary 4, $\Sigma_X = \sigma(\mathcal{F}_X)$, where $\mathcal{F}_X = \cup_{p\geq 1}\pi_p^{-1}\left(\Sigma_{X_p}\right)$ is the cylindrical field.

(i) Suppose there exists $p^* \geq 1$ such that $\overline{\mu}^{p^*}$ is mutually singular. Since the limit LPS $\overline{\mu}$ satifies $\widehat{\pi}_{p^*}(\overline{\mu}) = \overline{\mu}^{p^*}$, the result is an immediate consequence of Lemma 5.(1).

(ii) Let $\overline{\mu}$ be of full-support. Since each $\pi_{p,q}$ is a continuous surjection, so is each π_p. It follows from Lemma 7.(1) that $\widehat{\pi}_p(\overline{\mu}) = \overline{\mu}^p$ is of full-support, for each $p \geq 1$.

Conversely, assume that each $\overline{\mu}^p$ is of full-support. Let $B \subseteq X$ be a non-empty basic open set such that, by Proposition 11, $B = \pi_q^{-1}(O_q)$ where O_q is open in X_q, for some $q \geq 1$. Since $\overline{\mu}^q$ is of full-support, by Lemma 6 there exists $l \leq n$, such that $\mu_l^q(O_q) > 0$; consequently, it follows that

$$\begin{aligned} \mu_l(B) &= \mu_l\left(\pi_q^{-1}(O_q)\right) \\ &= \mu_l^q(O_q) \\ &> 0. \end{aligned}$$

Using again Lemma 6, we conclude that $\overline{\mu}$ is of full-support, as required. ■

5.1.4 Proof of Lemma 1.

(i): Let $\{(h_i)_n\}_{n\in\mathbb{N}} = \left\{\left(\overline{\mu}_i^1\right)_n, \left(\overline{\mu}_i^2\right)_n, ...\right\}_{n\in\mathbb{N}}$ be a sequence in H_i^1 converging in the product topology to $h_i^* = \left(\left(\overline{\mu}_i^1\right)^*, \left(\overline{\mu}_i^2\right)^*, ...\right)$, that is, $\left(\overline{\mu}_i^k\right)_n \to \left(\overline{\mu}_i^k\right)^*$ for each $k \geq 1$. We have to show that $h_i^* \in H_i^1$, i.e., $\overline{\mathrm{marg}}_{X_i^{k-1}}\left(\overline{\mu}_i^{k+1}\right)^* = \left(\overline{\mu}_i^k\right)^*$ for all $k \geq 1$. By Lemma 4, it holds that, for all $k \geq 1$, $\widehat{\mathrm{Proj}}_{X_i^{k-1}} : \mathcal{N}\left(X_i^k\right) \to \mathcal{N}\left(X_i^{k-1}\right)$ is a continuous function. Hence, for all $k \geq 1$, $\left(\overline{\mu}_i^k\right)_n \to \left(\overline{\mu}_i^k\right)^*$ implies $\overline{\mathrm{marg}}_{X_i^{k-1}}\left(\overline{\mu}_i^{k+1}\right)_n \to \overline{\mathrm{marg}}_{X_i^{k-1}}\left(\overline{\mu}_i^{k+1}\right)^*$, which proves the claim.

(ii): Since $\widetilde{\Lambda}_i^1 = \widetilde{\Lambda}_i^0 \cap H_i^1$ and, by the above, H_i^1 is closed in H_i^0, it suffices to show that $\widetilde{\Lambda}_i^0$ is Borel. If each space S_i is Polish, we will show that $\widetilde{\Lambda}_i^0$ is a G_δ-subset of H_i^0. By definition, $\widetilde{\Lambda}_i^0$ can be written as a countable union of cylinder sets, namely

$$\widetilde{\Lambda}_i^0 = \bigcup_{k\geq 1}\left\{h_i = \left(\overline{\mu}_i^1, \overline{\mu}_i^2, ...,\right) \in H_i^0 \,\middle|\, \overline{\mu}_i^k \in \mathcal{L}\left(X_i^{k-1}\right)\right\}.$$

It follows from Lemma 8.(2) that each set $\left\{h_i \in H_i^0 \left| \overline{\mu}_i^{k'} \in \mathcal{L}\left(X_i^{k'-1}\right)\right.\right\}$ is a Borel cylinder in H_i^0 with a G_δ base, hence a G_δ-subset of H_i^0 ([25, Exercise 2.3.B.(b)]). If each space of primitive uncertainty S_i is Polish, so is each H_i^0, and, by part (i), H_i^1 is also Polish. Each cylinder set $\left\{h_i \in H_i^0 \left| \overline{\mu}_i^{k'} \in \mathcal{L}\left(X_i^{k'-1}\right)\right.\right\}$ is Polish subspace of H_i^0, since, by the above, it is a G_δ-set in H_i^0. Thus $\widetilde{\Lambda}_i^0$ is a countable union of Polish subspaces of H_i^0, hence Polish (and so a G_δ-set) in H_i^0.

5.1.5 Proof of Lemma 2.

The family $\mathcal{P} = (Z_k, \pi_{k,k+1})_{k\geq 0}$ is a projective system of Polish (so metrizable Lusin) spaces, and each bonding map $\pi_{k,k+1} : Z_{k+1} \to Z_k$ is a coordinate projection. By standard arguments (see [45, pp. 116-117] or [46, p. 416]) it follows that the projective limit set is non-empty and it can be identified homeomorphically with the Cartesian product $Z = \prod_{l=0}^{\infty} W_l$. Thus the conclusion is immediate from Theorem 4.

5.2 Proofs for Section 3.4

5.2.1 Lexicographic type structures and self-evident events.

Here, we formalize the idea (mentioned in the main text) that a lexicographic type structure represents a set of restrictions on players' hierarchies of beliefs that are "transparent". This requires an epistemic apparatus, so we need to introduce further notations and terminology. The exposition follows mainly [3, Appendix A].

In what follows, let $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i\in I}$ an arbitrary lexicographic type structure. We say that a type $t_i \in T_i$ *believes* an event $E \subseteq S_{-i} \times T_{-i}$ if $\beta_i(t_i)(E) = \overrightarrow{1}$. (The same definition of belief was used in Section 3.3 with reference to hierarchies of lexicographic beliefs.) Given an event $E_{-i} \subseteq S_{-i} \times T_{-i}$, let

$$\mathbf{B}_i(E_{-i}) = S_i \times \left\{t_i \in T_i \left| \beta_i(t_i)(E_{-i}) = \overrightarrow{1}\right.\right\}.$$

Given events $E_i \subseteq S_i \times T_i$ for each $i \in I$, we write

$$\mathbf{B}\left(\prod_{i\in I} E_i\right) = \prod_{i\in I} \mathbf{B}_i(E_{-i}).$$

Using σ-additivity of probability measures, the following properties of the belief operator $\mathbf{B}$:$S \times T \rightrightarrows S \times T$ are easily verified.

- B1 **Monotonicity property**: For each $i \in I$, fix events $E_i, F_i \subseteq S_i \times T_i$. If $E_i \subseteq F_i$ for each $i \in I$, then $\mathbf{B}(\Pi_{i\in I} E_i) \subseteq \Pi_{i\in I} \mathbf{B}_i(F_{-i})$.
- B2 **Conjunction property**: For each $i \in I$, let $\{E_i^n\}_{n\in\mathbb{N}}$ be a sequence of events in $S_i \times T_i$. Thus
$$\cap_{n\in\mathbb{N}} \mathbf{B}(\Pi_{i\in I} E_i^n) = \mathbf{B}(\cap_{n\in\mathbb{N}}(\Pi_{i\in I} E_i^n)).$$

Definition 20 *The event* $\Pi_{i\in I} E_i \subseteq S \times T$ *is* ***self-evident (in*** $\mathcal{T}$***)*** *if* $\Pi_{i\in I} E_i \subseteq \mathbf{B}(\Pi_{i\in I} E_i)$.

We define also common belief operator $\mathbf{CB}: S\times T \rightrightarrows S\times T$ as follows. For each player $i\in I$, fix events $E_i \subseteq S_i \times T_i$. We iterate the belief operator $\mathbf{B}$ as follows:

$$\begin{aligned}\mathbf{B}^0\left(\Pi_{i\in I}E_i\right) &= \Pi_{i\in I}E_i,\\ \mathbf{B}^{k+1}\left(\Pi_{i\in I}E_i\right) &= \mathbf{B}\left(\mathbf{B}^k\left(\Pi_{i\in I}E_i\right)\right),\ \ \forall k\geq 0.\end{aligned}$$

So, let

$$\mathbf{CB}\left(\Pi_{i\in I}E_i\right) = \cap_{k\geq 0}\mathbf{B}^k\left(\Pi_{i\in I}E_i\right).$$

Lemma 13 *For each $i\in I$, fix events $E_i\subseteq S_i\times T_i$. Thus $\Pi_{i\in I}E_i$ is self-evident (in $\mathcal{T}$) if and only if $\Pi_{i\in I}E_i = \mathbf{CB}\left(\Pi_{i\in I}E_i\right)$.*

The following result establishes the connection between the notion of self-evident event and that of type morphism.

Proposition 12 *Fix a lexicographic type structure $\mathcal{T}' = \langle S_i, T_i', \beta_i'\rangle_{i\in I}$.*

(i) *If $(\varphi_i)_{i\in I}: T\to T'$ is a bimeasurable type morphism from $\mathcal{T} = \langle S_i, T_i, \beta_i\rangle_{i\in I}$ to $\mathcal{T}'$, then $S\times \Pi_{i\in I}\varphi_i(T_i)$ is a self-evident event in $\mathcal{T}'$.*

(ii) *Let $S\times\Pi_{i\in I}E_i'\subseteq S\times T'$ be self-evident in $\mathcal{T}'$. For each $i\in I$, let $\varphi_i: E_i'\to T_i'$ be the identity map. Thus there exists a lexicographic type structure $\mathcal{T} = \langle S_i, E_i', \beta_i\rangle_{i\in I}$ such that $(\varphi_i)_{i\in I}: \Pi_{i\in I}E_i' \to T'$ is a bimeasurable type morphism from $\mathcal{T}$ to $\mathcal{T}'$.*

Proof: Part (i): By bimeasurability, each set $\varphi_i(T_i)$ is Lusin subspace of T_i', so $S\times \Pi_{i\in I}\varphi_i(T_i)$ is Borel in $S\times T'$. We need to show that $S\times\Pi_{i\in I}\varphi_i(T_i)\subseteq \mathbf{B}\left(S\times\Pi_{i\in I}\varphi_i(T_i)\right)$. This will be accomplished by showing that, for each $\varphi_i(t_i)\in\varphi_i(T_i)$,

$$\beta_i'\left(\varphi_i(t_i)\right)\left(S_{-i}\times\varphi_{-i}(T_{-i})\right) = \overrightarrow{1}.$$

But this follows immediately from the definition of type morphism, indeed

$$\begin{aligned}\beta_i'\left(\varphi_i(t_i)\right)\left(S_{-i}\times\varphi_{-i}(T_{-i})\right) &= \beta_i(t_i)\left(S_{-i}\times T_{-i}\right)\\ &= \overrightarrow{1}.\end{aligned}$$

Part (ii): We construct a type structure $\mathcal{T} = \langle S_i, T_i, \beta_i\rangle_{i\in I}$ as follows. For each $i\in I$, set $T_i = E_i'$. Since each E_i' is Borel in T_i', then E_i' is Lusin metrizable in the relative topology, hence T_i is Lusin metrizable space in its own right. Furthermore, the Borel σ-field on $S_{-i}\times T_{-i}$ is the one inherited from the Borel σ-field on $S_{-i}\times T_{-i}'$. Thus we can define each belief map β_i as $\beta_i(t_i)(F_{-i}) = \beta_i'(t_i)(F_{-i})$, for any event $F_{-i}\subseteq S_{-i}\times T_{-i}$. For each $t_i\in T_i$, $\beta_i(t_i)$ is a well-defined LPS over $S_{-i}\times T_{-i}$, in that

$$\begin{aligned}\beta_i(t_i)\left(S_{-i}\times T_{-i}\right) &= \beta_i'(t_i)\left(S_{-i}\times E_{-i}'\right)\\ &= \overrightarrow{1},\end{aligned}$$

where the first equality is by definition and the fact that $E_{-i}' = T_{-i}$, while the second equality follows from the fact that $S\times\Pi_{i\in I}E_i'$ is self-evident in $\mathcal{T}'$. We now show that each belief map

β_i is measurable; by this, it will follow that $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i\in I}$ is a well-defined type structure. Since $E'_{-i} = T_{-i}$ is an event in T'_{-i}, then by Lemma 9 there exists a homeomorphism

$$\vartheta : \mathcal{N}\left(S_{-i} \times T_{-i}\right) \to \left\{\overline{\mu}_i \in \mathcal{N}\left(S_{-i} \times T'_{-i}\right) \middle| \overline{\mu}_i \left(S_{-i} \times T_{-i}\right) = \overrightarrow{1}\right\}.$$

Hence, for each Borel $G_{-i} \subseteq \mathcal{N}(S_{-i} \times T_{-i})$ the set

$$\left(\beta'_i\right)^{-1}\left(\vartheta\left(G_{-i}\right)\right) = \left\{t'_i \in T'_i \left|\beta'_i\left(t'_i\right) \in \vartheta\left(G_{-i}\right)\right.\right\}$$

is Borel in T'_{-i}. By the property of ϑ, it follows that

$$\begin{aligned}\left(\beta'_i\right)^{-1}\left(\vartheta\left(G_{-i}\right)\right) \cap E'_i &= \left\{t_i \in T_i \left|\beta_i\left(t_i\right) \in G_{-i}\right.\right\} \\ &= \beta_i^{-1}\left(G_{-i}\right).\end{aligned}$$

I.e., $\beta_i^{-1}\left(G_{-i}\right)$ is measurable in T_i, as it is the intersection of two measurable sets.

Finally, it remains to show that $\left(\varphi_i\right)_{i\in I} : \Pi_{i\in I} E'_i \to T'$ is a bimeasurable type morphism from $\mathcal{T}$ to $\mathcal{T}'$. Since each $\varphi_i : E'_i \to T'_i$ is the identity map, $\left(\varphi_i\right)_{i\in I}$ is bimeasurable (in fact, a measure-theoretic isomorphism). Clearly, it is immediate to check that each identity map $\varphi_i : E'_i \to T'_i$ is such that

$$\beta'_i \circ \varphi_i = \left(\widehat{Id_{S_{-i}}, \varphi_{-i}}\right) \circ \beta_i.$$

Thus, $\left(\varphi_i\right)_{i\in I}$ is a bimeasurable type morphism from $\mathcal{T}$ to $\mathcal{T}'$, as required. ■

5.2.2 Proof of Theorem 1

The proof is divided in two main steps. In the first step, we show that for each $t_i \in T_i$, the corresponding *i-description* $d_i\left(t_i\right)$ belongs to H_i, the collection of infinite hierarchies of LPS's satisfying collective coherence. In the second step, we show that the map $\left(d_i\right)_{i\in I}$ is a type morphism. We do not show the uniqueness of type morphism $\left(d_i\right)_{i\in I}$ since this follows from routine arguments (cf. [4] or [29]). In both cases, the proof is by induction.

First step: $d_i\left(T_i\right) \subseteq H_i$. By definition of i-description, $d_i\left(T_i\right) \subseteq H_i^0$. We use induction to prove $d_i\left(T_i\right) \subseteq H_i$.

(*Base step*): We first show that $d_i\left(T_i\right) \subseteq H_i^1$, so we need to verify that for all $t_i \in T_i$ and all $k \geq 1$,

$$\overline{\operatorname{marg}}_{X_i^{k-1}}\left(d_i^{k+1}\left(t_i\right)\right) = d_i^k\left(t_i\right),$$

that is,

$$\widehat{\operatorname{Proj}}_{X_i^{k-1}}\left(d_i^{k+1}\left(t_i\right)\right) = d_i^k\left(t_i\right). \tag{5.2}$$

(recall that $\operatorname{Proj}_{X_i^{k-1}}$ stands for the coordinate projection from $X_i^k = X_i^{k-1} \times \mathcal{N}\left(X_{-i}^{k-1}\right)$ onto X_i^{k-1} and d_i^{k+1} is a map from T_i into $\mathcal{N}\left(X_i^k\right) = \mathcal{N}\left(X_i^{k-1} \times \mathcal{N}\left(X_{-i}^{k-1}\right)\right)$). To this end, pick any event $E_{k-1} \in \Sigma_{X_i^{k-1}}$. Thus

$$\begin{aligned}
\widehat{\mathrm{Proj}}_{X_i^{k-1}} \left(d_i^{k+1}(t_i) \right) (E_{k-1}) &= d_i^{k+1}(t_i) \left(\mathrm{Proj}^{-1}_{X_i^{k-1}} (E_{k-1}) \right) \\
&= \widehat{\psi}^k_{-i} (\beta_i (t_i)) \left(E_{k-1} \times \mathcal{N}\left(X_{-i}^{k-1} \right) \right) \\
&= \beta_i (t_i) \left(\left\{ (s_{-i}, t_{-i}) \middle| \psi^k_{-i} (s_{-i}, t_{-i}) \in E_{k-1} \times \mathcal{N}\left(X_{-i}^{k-1} \right) \right\} \right) \\
&= \beta_i (t_i) \left(\left\{ (s_{-i}, t_{-i}) \middle| \begin{array}{c} \psi^{k-1}_{-i} (s_{-i}, t_{-i}) \in E_{k-1}, \\ d^k_{-i} (t_{-i}) \in \mathcal{N}\left(X_{-i}^{k-1} \right) \end{array} \right\} \right) \\
&= \beta_i (t_i) \left(\left\{ (s, t_{-i}) \middle| \psi^{k-1}_{-i} (s_{-i}, t_{-i}) \in E_{k-1} \right\} \right) \\
&= \widehat{\psi}^{k-1}_{-i} (\beta_i (t_i)) (E_{k-1}) \\
&= d_i^k (t_i) (E_{k-1}),
\end{aligned}$$

where the fourth equality follows from the definition of ψ^k_{-i}, and the fifth equality follows from $d^k_{-i}(T_{-i}) \subseteq \mathcal{N}\left(X_{-i}^{k-1} \right)$. So, Eq. (5.2) is proved.

To continue the proof, we need the following

Claim 4 *For each $i \in I$, let f_i be the homeomorphism of Proposition 2. Thus, the following diagram commutes:*

$$\begin{array}{ccc}
T_i & \xrightarrow{\beta_i} & \mathcal{N}(S_{-i} \times T_{-i}) \\
\Big\downarrow d_i & & \Big\downarrow \widehat{(Id_{S_{-i}}, d_{-i})} \\
H_i^1 & \xrightarrow{f_i} & \mathcal{N}(S_{-i} \times H_{-i}^0)
\end{array} \tag{5.3}$$

Proof of Claim: We will show that for each $k \geq 0$,

$$\overline{\mathrm{marg}}_{X_i^k} \left(\widehat{Id_{S_{-i}}, d_{-i}} \right) (\beta_i (t_i)) = d_i^{k+1} (t_i).$$

By property of f_i, $\overline{\mathrm{marg}}_{X_i^k} f_i(d_i(t_i)) = d_i^{k+1}(t_i)$ (cf. Lemma 2). By Lemma 2, for each $h_i = (\overline{\mu}_i^1, \overline{\mu}_i^2, ...) \in H_i^1$, there exists a *unique* $\overline{\mu}_i \in \mathcal{N}(S_{-i} \times H_{-i}^0)$ such that, for each $k \geq 0$, $\overline{\mathrm{marg}}_{X_i^k} \overline{\mu}_i = \overline{\mu}_i^k$. Thus, it must hold that

$$f_i(d_i(t_i)) = \left(\widehat{Id_{S_{-i}}, d_{-i}} \right) (\beta_i (t_i)).$$

Fix $E_k \in \Sigma_{X_i^k}$. We have

$$\begin{aligned}
\overline{\mathrm{marg}}_{X_i^k} \left(\widehat{Id_{S_{-i}}, d_{-i}} \right) (\beta_i (t_i)) (E_k) &= \left(\widehat{Id_{S_{-i}}, d_{-i}} \right) (\beta_i (t_i)) \left(\mathrm{Proj}^{-1}_{X_i^k} (E_k) \right) \\
&= \beta_i (t_i) \left(\left(Id_{S_{-i}}, d_{-i} \right)^{-1} \left(\mathrm{Proj}^{-1}_{X_i^k} (E_k) \right) \right) \\
&= \beta_i (t_i) \left(\left\{ (s_{-i}, t_{-i}) \middle| (s_{-i}, d_{-i} (t_{-i})) \in \mathrm{Proj}^{-1}_{X_i^k} (E_k) \right\} \right) \\
&= \beta_i (t_i) \left(\left\{ (s_{-i}, t_{-i}) \middle| \left(s_{-i}, d^1_{-i} (t_{-i}), ..., d^k_{-i} (t_{-i}) \right) \in E_k \right\} \right) \\
&= \beta_i (t_i) \left(\left(\psi^k_{-i} \right)^{-1} (E_k) \right) \\
&= \widehat{\psi}^k_{-i} (\beta_i (t_i)) (E_k) \\
&= d_i^{k+1} (t_i) (E_k).
\end{aligned}$$

■

(*Inductive step*): Recall that $d_i(t_i) \in H_i^l$, $l \geq 2$, if and only if $f_i(d_i(t_i))\left(S_{-i} \times H_{-i}^{l-1}\right) = \overrightarrow{1}$, for each $t_i \in T_i$. Suppose that, for each player $i \in I$, $d_i(T_i) \subseteq H_i^{l-1}$. Hence, for all $t_i \in T_i$:

$$\begin{aligned} f_i(d_i(t_i))\left(S_{-i} \times H_{-i}^{l-1}\right) &= \left(\widehat{Id_{S_{-i}}, d_{-i}}\right)(\beta_i(t_i))\left(S_{-i} \times H_{-i}^{l-1}\right) \\ &= \beta_i(t_i)\left(\left(Id_{S_{-i}}, d_{-i}\right)^{-1}\left(S_{-i} \times H_{-i}^{l-1}\right)\right) \\ &= \beta_i(t_i)\left(\left\{(s_{-i}, t_{-i}) : d_{-i}(t_{-i}) \in H_{-i}^{l-1}\right\}\right) \\ &= \beta_i(t_i)(S_{-i} \times T_{-i}) \\ &= \overrightarrow{1}, \end{aligned}$$

where the first equality follows from Claim 4 and the fourth from the induction hypothesis. Thus $f_i(d_i(t_i))\left(S_{-i} \times H_{-i}^{l-1}\right) = \overrightarrow{1}$, as required.

Second step: $(d_i)_{i \in I}$ *is a type morphism from* $\mathcal{T}$ *to* $\mathcal{T}_u$. First, we show that $(d_i)_{i \in I}$ is measurable. Since $Id_{S_{-i}}$ is continuous (hence measurable), we need to show - by induction - that $d_i = \left(d_i^1, d_i^2, ...\right)$ is measurable, for each $i \in I$. By definition, $d_i^1 = \widehat{\mathrm{Proj}}_{S_{-i}} \circ \beta_i$, where β_i is measurable by assumption, and $\widehat{\mathrm{Proj}}_{S_{-i}}$ is measurable (in fact, continuous) by Lemma 4. Hence d_i^1 is measurable, for each $i \in I$. Now assume, by way of induction, that for $i \in I$, $k = 1, ...l$, d_i^k is measurable. This implies that $\psi_{-i}^l = \left(Id_{S_{-i}}, d_{-i}^1, ..., d_{-i}^l\right)$ is also measurable. Then, by Lemma 4, the map $\widehat{\psi}_{-i}^l$ is measurable and thus $d_i^{l+1} = \widehat{\psi}_{-i}^l \circ \beta_i$ is also measurable. Finally, note that, since $d_i(T_i) \subseteq H_i$ for each $i \in I$ (as proved in the first step), it follows from Proposition 3 and Diagram (5.3) that

$$f_i \circ d_i = \widehat{\psi}_{-i} \circ \beta_i,$$

which implies that the conditions of Definition 9 are met. Hence $(d_i)_{i \in I}$ is a type morphism, as required. ■

5.2.3 Proof of Proposition 6

If each type space T_i is countable (e.g. finite), then so is $d_i(T_i)$, hence Borel in H_i. If instead $\mathcal{T}$ is non-redundant, then the map $(d_i)_{i \in I} : T \to H$ turns out to be a measure-theoretic isomorphism onto its image by Souslin Theorem. Thus, in both cases, the type morphism $(d_i)_{i \in I}$ is bimeasurable, and the conclusion follows from Proposition 12.(i).

On the other hand, let $S \times \Pi_{i \in I} E_i$ be self-evident in $\mathcal{T}_u$. By Proposition 12.(ii), there exists a lexicographic type structure $\mathcal{T} = \langle S_i, E_i, \beta_i' \rangle_{i \in I}$ such that the identity map $(\varphi_i)_{i \in I} : \Pi_{i \in I} E_i \to T$.is a bimeasurable type morphism from $\mathcal{T}$ to $\mathcal{T}_u$. Thus, it is easily verified that $(\varphi_i)_{i \in I} = (d_i)_{i \in I}$ is a type isomorphism and since $\mathcal{T}_u$ is non-redundant, $\mathcal{T}$ is also non-redundant. ■

5.3 Proofs for Section 3.7

5.3.1 Additional details of Example 1

Here, we provide a complete proof that the hierarchy induced by Player 1's type t_1' in the type structure described in Example 1 is not a mutually singular hierarchy, formally $d_1(t_1') = \left(d_1^1(t_1'), d_1^2(t_1'), ...\right) \notin \widetilde{\Lambda}_1$. The proof, which is by induction, makes use of the following Claim:

Claim 5 *For all $k \geq 1$, $\sigma\left(\psi_2^k\right) = 2^{S_2} \times \{\emptyset, T_2\}$.*

In the proof of Claim 5 we use the following well-known mathematical fact: Fix measurable spaces X_1, X_2, Y_1 and Y_2. Let $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ be measurable maps such that $\Sigma_{X_1} = \sigma(f_1)$ and $\Sigma_{X_2} = \sigma(f_2)$. Thus $\Sigma_{X_1} \times \Sigma_{X_2} = \sigma((f_1, f_2))$.

Proof of Claim 5: First note that, since $\beta_2(t_2') = \beta_2(t_2'')$, types t_2' and t_2'' obviously induce the same hierarchy for Player 2. Hence $d_2(t_2') = d_2(t_2')$, so that $d_2^k(t_2') = d_2^k(t_2')$ for all $k \geq 1$. Each d_2^k is a constant map, so $\sigma\left(d_2^k\right) = \{\emptyset, T_2\}$ for all $k \geq 1$. By definition, $\psi_2^k = \left(Id_{S_2}, d_2^1, ..., d_2^{k-1}, d_2^k\right)$ for all $k \geq 1$. Since $\sigma(Id_{S_2}) = 2^{S_2}$, we obtain from the above mentioned fact that $\sigma\left(\psi_2^k\right) = 2^{S_2} \times \{\emptyset, T_2\}$ for all $k \geq 1$, as required. ■

The base step, i.e., $d_1^1(t_1') = \overline{\mathrm{marg}}_{S_2}(\beta_1(t_1'))$ is not a mutually singular LPS, was already shown in Example 1. Suppose that $d_1^k(t_i) = \widehat{\psi}_2^{k-1}(\beta_1(t_1'))$ is not mutually singular for $k \geq 1$. Using (the contrapositive of) Lemma 5.(2) we deduce that there are no Borel sets $E_1, E_2 \in \sigma\left(\psi_2^{k-1}\right)$ satisfying the requirement of mutual singularity for $\beta_1(t_1')$. But $\sigma\left(\psi_2^k\right) = 2^{S_2} \times \{\emptyset, T_2\}$ by Claim 5, hence, using again Lemma 5.(2), we conclude that $d_1^{k+1}(t_i) = \widehat{\psi}_2^k(\beta_1(t_1'))$ is not mutually singular.

5.3.2 Proof of Proposition 7.

The proof follows the lines of the proof of Lemma 6.2 in [28], and we shall only indicate the additional needed arguments.

To this end, we need to recast our analysis in a purely measurable framework. We first provide a definition of lexicographic type structure which relies only on measure-theoretic concepts, without any reference to the topology on each type space. Recall that, given a measurable space (X, Σ_X), the set $\mathcal{N}(X)$ is endowed with the σ-field $\mathcal{A}_{\mathcal{N}(X)}$ generated by sets of the form

$$\{(\mu_1, ..., \mu_n) \in \mathcal{N}(X) \,|\, \mu_l(E) \geq p_l, \forall l \leq n\},$$

where $E \in \Sigma_X$ and $p_l \in \mathbb{Q} \cap [0, 1]$ for all $l \leq n$. If $\mathcal{F}_X$ is a field generating Σ_X, then $\mathcal{A}_{\mathcal{N}(X)}$ is generated by sets of the form

$$\{(\mu_1, ..., \mu_n) \in \mathcal{N}(X) \,|\, \mu_l(F) \geq p_l, \forall l \leq n\},$$

where $F \in \mathcal{F}_X$ and $p_l \in \mathbb{Q} \cap [0, 1]$ for all $l \leq n$. (Corollary 3 in Appendix 5.1.2.)

Definition 21 *A* ***measurable*** $(S_i)_{i \in I}$***-based lexicographic type structure*** *is a structure* $\mathcal{T} = \langle S_i, (T_i, \Sigma_{T_i}), \beta_i \rangle_{i \in I}$*, where*

1. *for each $i \in I$, (T_i, Σ_{T_i}) is a measurable space;*

2. *for each $i \in I$, the function $\beta_i : T_i \to \mathcal{N}(S_{-i} \times T_{-i})$ is measurable.*

Definition 6 in the main text is a special case of a measurable type structure.

Next note that, under our topological assumptions, it turns out that for each player i, the product σ-field over the hierarchy space $H_i^0 = \prod_{k=0}^{\infty} \mathcal{N}\left(X_i^k\right)$ coincides with the Borel σ-field generated by the product topology—this follows from the fact that each $\mathcal{N}\left(X_i^k\right)$ is metrizable Lusin, so second countable, and from [1, Theorem 4.44]. The family of Borel cylinders in H_i^0 is a field which generates the Borel σ-field $\Sigma_{H_i^0}$. Thus, by Corollary 3 in Appendix 5.1.2, the Borel σ-field over the space $\mathcal{N}\left(S_{-i} \times H_{-i}^0\right)$ is generated by sets of the form

$$\left\{ \left(\mu_i^1, ..., \mu_i^n\right) \in \mathcal{N}\left(S_{-i} \times H_{-i}^0\right) \middle| \mu_i^l \left(F_{-i}^m \times \prod_{k=m}^{\infty} \mathcal{N}\left(X_{-i}^k\right)\right) \geq p_l, \forall l \leq n \right\},$$

where F_{-i}^m is a Borel subset of $S_{-i} \times \prod_{k=0}^{m-1} \mathcal{N}\left(X_{-i}^k\right)$ and $(p_1, p_2, ..., p_n) \in (\mathbb{Q} \cap [0,1])^n$. So, it makes sense to abstract from the topological aspects of the construction by relying only to the "natural" σ-field $\Sigma_{H_i^0}$ on each H_i^0. With this, for any measurable type structure $\mathcal{T} = \langle S_i, (T_i, \Sigma_{T_i}), \beta_i \rangle_{i \in I}$, the notion of each hierarchy desciption map $d_i : T_i \to H_i^0$ (as stated in the main text) is well-defined in a topology-free framework, i.e., d_i is measurable with respect to Σ_{T_i}.

Having done these preparations, we first show that $\Pi_{i \in I} \sigma(d_i) \subseteq \Pi_{i \in I} \mathcal{G}_{T_i}$. Fix a measurable type structure $\mathcal{T} = \langle S_i, (T_i, \Sigma_{T_i}), \beta_i \rangle_{i \in I}$; we now construct a measurable type structure $\mathcal{T}' = \langle S_i, (T_i, \mathcal{G}_{T_i}), \beta_i' \rangle_{i \in I}$ such that

$$\beta_i'(t_i)(E) = \beta_i(t_i)(E), \ \ \forall t_i \in T_i, \forall E \in \Sigma_{S_{-i}} \times \mathcal{G}_{T_{-i}}.$$

Since $\Pi_{i \in I} \mathcal{G}_{T_i}$ is closed under $\mathcal{T}$, this implies that each β_i' is $\mathcal{G}_{T_i}$-measurable, and so $\mathcal{T}'$ is a well-defined measurable type structure. We show that $\mathcal{T}$ and $\mathcal{T}'$ induce the same hierachies of lexicographic beliefs. That is, we show that $d_i(t_i) = d_i'(t_i)$ for each $t_i \in T$ and $i \in I$. (Here, d_i' denotes the hierarchy map associated with $\mathcal{T}'$.) This will entail $\Pi_{i \in I} \sigma(d_i) = \Pi_{i \in I} \sigma(d_i')$, and since each d_i' is measurable, then $\Pi_{i \in I} \sigma(d_i') \subseteq \Pi_{i \in I} \mathcal{G}_{T_i}$, as required.

Since $\overline{\mathrm{marg}}_{S_{-i}} \beta_i'(t_i) = \overline{\mathrm{marg}}_{S_{-i}} \beta_i(t_i)$ for all $t_i \in T_i$, this immediately yields $d_i^1 = \left(d_i^1\right)'$, i.e., the first-order hierarchy description maps coincide. Next, suppose that the statement holds true for $k \geq 1$, i.e., $d_i^k = \left(d_i^k\right)'$ for each $i \in I$; this in turn implies that $\psi_{-i}^k = \left(s_{-i}, d_{-i}^1, ..., d_{-i}^k,\right) = \left(s_{-i}, \left(d_i^1\right)', ..., \left(d_i^k\right)'\right) = \left(\psi_{-i}^k\right)'$ holds true. Then, for each $E_i^k \in \Sigma_{X_i^k}$,

$$d_i^{k+1}(t_i)\left(E_i^k\right) = \widehat{\psi}_{-i}^k\left(\beta_i(t_i)\right)\left(E_i^k\right) = \widehat{\left(\psi_{-i}^k\right)'}\left(\beta_i'(t_i)\right)\left(E_i^k\right) = \left(d_i^{k+1}\right)'(t_i)\left(E_i^k\right),$$

as required.

To prove that $\Pi_{i \in I} \mathcal{G}_{T_i} \subseteq \Pi_{i \in I} \sigma(d_i)$, we need to show that $\Pi_{i \in I} \sigma(d_i)$ is closed under $\mathcal{T}$. Fix $(p_1, p_2, ..., p_n) \in (\mathbb{Q} \cap [0,1])^n$, and let F_{-i}^m be a measurable subset of $S_{-i} \times \prod_{k=0}^{m-1} \mathcal{N}\left(X_{-i}^k\right) = X_i^m$. Define the following set:

$$B = (\beta_i)^{-1}\left(\left\{\left(\mu_i^1, ..., \mu_i^n\right) \in \mathcal{N}\left(S_{-i} \times T_{-i}\right) \middle| \widetilde{\psi}_{-i}\left(\mu_i^l\right)\left(F_{-i}^m \times \prod_{k=m}^{\infty} \mathcal{N}\left(X_{-i}^k\right)\right) \geq p_l, \forall l \leq n\right\}\right);$$

we need to show that $B \in \sigma(d_i)$. To accomplish this task, we prove that

$$B = \left(d_i^m\right)^{-1}\left(\left\{\left(\mu_i^{m,1}, ..., \mu_i^{m,n}\right) \in \mathcal{N}\left(X_i^{m-1}\right) \middle| \mu_i^{m,l}\left(F_{-i}^m\right) \geq p_l, \forall l \leq n\right\}\right) \tag{5.4}$$

Indeed, if Eq. (5.4) holds, we can conclude: By definition of $\sigma(d_i)$, the map $d_i : T_i \to H_i^0$ is measurable, and this implies that d_i^m is measurable for all $m \geq 1$. So, if Eq. (5.4) holds, then the LHS of Eq. (5.4) is contained in $\sigma(d_i)$, establishing the claim.

Let $t_i \in T_i$ belong to the LHS of Eq. (5.4). Thus $t_i \in T_i$ is associated with length-n LPS, namely $\beta_i(t_i) = \left(\mu_i^1, ..., \mu_i^n\right) \in \mathcal{N}\left(S_{-i} \times T_{-i}\right)$, and the induced $(m+1)$-order LPS

$\left(\mu_i^{m+1,1},...,\mu_i^{m+1,n}\right)=\left(\widetilde{\psi}_{-i}^m\left(\mu_i^1\right),...,\widetilde{\psi}_{-i}^m\left(\mu_i^n\right)\right)$ is such that, for all $l\leq n$,

$$\begin{aligned}
\mu_i^{m+1,l}\left(F_{-i}^m\right) &= \mu_i^l\left(\left(\psi_{-i}^m\right)^{-1}\left(F_{-i}^m\right)\right)\\
&= \mu_i^l\left(\left(\psi_{-i}\right)^{-1}\left(\mathrm{Proj}_{X_i^m}^{-1}\left(F_{-i}^m\right)\right)\right)\\
&= \mu_i^l\left(\left(\psi_{-i}\right)^{-1}\left(F_{-i}^m\times\prod_{k=m}^{\infty}\mathcal{N}\left(X_i^k\right)\right)\right)\\
&\geq p_l,
\end{aligned}$$

where the second equality follows from the fact that $\mathrm{Proj}_{X_i^m}\circ\psi_{-i}=\psi_{-i}^m$, for all $m\in\mathbb{N}$. This shows that $t_i\in T_i$ belongs to the RHS of Eq. (5.4). Conversely, suppose that $t_i\in T_i$ belongs to the RHS of Eq. (5.4). Note that

$$\begin{aligned}
d_i^m\left(t_i\right)\left(F_{-i}^m\right) &= \overline{\mathrm{marg}}_{X_i^m}\widehat{\psi}_{-i}\left(\beta_i\left(t_i\right)\right)\left(F_{-i}^m\right)\\
&= \widehat{\psi}_{-i}\left(\beta_i\left(t_i\right)\right)\left(F_{-i}^m\times\prod_{k=m}^{\infty}\mathcal{N}\left(X_{-i}^k\right)\right)\\
&= \begin{pmatrix}\mu_i^1\left(\left(\psi_{-i}\right)^{-1}\left(F_{-i}^m\times\prod_{k=m}^{\infty}\mathcal{N}\left(X_{-i}^k\right)\right)\right),...,\\ \mu_i^n\left(\left(\psi_{-i}\right)^{-1}\left(F_{-i}^m\times\prod_{k=m}^{\infty}\mathcal{N}\left(X_{-i}^k\right)\right)\right)\end{pmatrix}\\
&= \left(\mu_i^{m,1}\left(F_{-i}^m\right),...,\mu_i^{m,n}\left(F_{-i}^m\right)\right),
\end{aligned}$$

hence the conclusion that $t_i\in T_i$ also belongs to the LHS of Eq. (5.4) is immediate.

5.3.3 Proof of Proposition 8.

Given a mutually singular type structure $\mathcal{T}=\langle S_i,T_i,\beta_i\rangle_{i\in I}$, we show that a type $t_i\in T_i$ induces a hierarchy with a mutually singular representation, i.e., $d_i\left(t_i\right)\in\Lambda_i^1$, if and only if $\beta_i\left(t_i\right)$ is mutually singular w.r.to $\sigma\left(\psi_{-i}\right)=\Sigma_{S_{-i}}\times\sigma\left(d_{-i}\right)$. The conclusion will follow from Proposition 7, according to which $\Sigma_{S_{-i}}\times\sigma\left(d_{-i}\right)=\Sigma_{S_{-i}}\times\mathcal{G}_{T_{-i}}$.

If $d_i\left(t_i\right)\in\Lambda_i^1$ for each $t_i\in T_i$, then by Theorem 1 and the definition of the map $g_i:\Lambda_i^1\to\mathcal{L}\left(S_{-i}\times H_{-i}\right)$ (which is a homeomorphism) the following diagram commutes:

$$\begin{array}{ccc}
T_i & \xrightarrow{\beta_i} & \mathcal{L}(S_{-i}\times T_{-i})\\
\downarrow{\scriptstyle d_i} & & \downarrow{\scriptstyle (\widehat{Id_{S_{-i}},d_{-i}})}\\
\Lambda_i^1 & \xrightarrow{g_i} & \mathcal{L}(S_{-i}\times H_{-i})
\end{array}$$

So there are measurable sets $\{E_l\}_{l=1}^n\subseteq\Sigma_{S_{-i}}\times\Sigma_{H_{-i}}$, $n\in\mathbb{N}$, which satisfy the requirement of mutual singularity for LPS $g_i\left(d_i\left(t_i\right)\right)$. The collection $\left\{\left(\psi_{-i}\right)^{-1}\left(E_l\right)\right\}_{l=1}^n$ belongs to the σ-field $\sigma\left(\psi_{-i}\right)=\Sigma_{S_{-i}}\times\sigma\left(d_{-i}\right)$, and such sets satisfy the desired properties of mutual singularity of $\beta_i\left(t_i\right)$.

On the other hand, suppose that each $\beta_i\left(t_i\right)$ is mutually singular w.r.to $\sigma\left(\psi_{-i}\right)$. It follows from the definition of $\sigma\left(\psi_{-i}\right)$ that there are pairwise disjoint, measurable sets $\{E_l\}_{l=1}^n\subseteq\Sigma_{S_{-i}}\times\Sigma_{H_{-i}}$, $n\in\mathbb{N}$, such that $\left(\beta_i\left(t_i\right)\right)_l\left(\left(\psi_{-i}\right)^{-1}\left(E_l\right)\right)=1$ and $\left(\beta_i\left(t_i\right)\right)_l\left(\left(\psi_{-i}\right)^{-1}\left(E_m\right)\right)=0$, for $l\neq m$. This means that $\widehat{\psi}_{-i}\left(\beta_i\left(t_i\right)\right)\in\mathcal{L}\left(S_{-i}\times H_{-i}^0\right)$, and since $\widehat{\psi}_{-i}\left(\beta_i\left(t_i\right)\right)=\overline{f}_i\left(d_i\left(t_i\right)\right)$ (Theorem 1), it follows from definition of Λ_i^1 that $d_i\left(t_i\right)\in\Lambda_i^1$.

5.3.4 Proof of Theorem 2.

The proof follows the lines of the proof of Theorem 1. For the reader's convenience, we provide only the necessary changes to be made.

The first step is to show by induction that, for each player $i \in I$, $d_i(t_i) \in \Lambda_i$. By Theorem 1 and Proposition 8, it follows that $d_i(t_i) \in \Lambda_i^1$. This in turn implies that the following diagram commutes:

$$
\begin{array}{ccc}
T_i & \xrightarrow{\beta_i} & \mathcal{L}(S_{-i} \times T_{-i}) \\
\Big\downarrow {\scriptstyle d_i} & & \Big\downarrow {\scriptstyle (\widehat{Id_{S_{-i}}, d_{-i}})} \\
\Lambda_i^1 & \xrightarrow{f_i} & \mathcal{L}(S_{-i} \times H_{-i})
\end{array}
\tag{5.5}
$$

To show that $d_i(t_i) \in \Lambda_i^l$, $l \geq 2$, one proceeds exactly as in the proof of Theorem 1, with the symbol H replaced by Λ, but making use this time of the commutativity of Diagram (5.5).

Having proved that $d_i(T_i) \subseteq \Lambda_i$ for each $i \in I$, by virtue of Proposition 4 and Diagram (5.5) we get

$$g_i \circ d_i = \widehat{\psi}_{-i} \circ \beta_i,$$

which shows that $(d_i)_{i \in I}$ is a type morphism.

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