Theoretical Computer Science, 2015-2016 **20/5/2016**
Sample exam. Time limit: 210 minutes.

This exam contains 4 questions. The total of all points is 14. There are 2 bonus points. You can use printed lecture materials and handwritten notes. You have a laptop at your disposal with all PDF-files of books and lecture notes used in class. You can make references to any of these materials. You can also make references to claims in exercises. You are not allowed to use the laptop for any other purpose.

Note: Unlike here, there will be at most 1 problem from the lecture notes ($\rightarrow$ exercise 2 and 3). On the exam you need to: answer true or false questions, prove some problem is undecidable, prove some problem is NP-complete, and some additional question.

1. Are the following statements true or false? Prove or disprove. (5)
   (a) The languages $\{a^m b^n : n - m \geq 2\}$, $\{a^m b^n : n + m \geq 2\}$, and the complement of $(a|bb)^*(b|aa)^*$ are regular.
   (b) Consider a non-deterministic automaton with states Q and accept states F that recognizes a language L. If we replace the accept states F by $Q \setminus F$, then the automaton recognizes the complement of L.
   (c) There exists a constant c such that every finite language can be decided by a Turing machine M that on input x computes for at most $c(|x| + 1)$ steps.
   (d) A subset of a countable set is also countable.
   (e) A subset of an enumerable set is also enumerable.
   (f) The following language is decidable:
   $$\{\langle M \rangle : M \text{ is a Turing machine, } \forall x\colon M(x) \text{ is defined and } \forall x\colon |M(x)| \leq |x|\}.$$
   (g) The following language is decidable:
   $$\left\{\langle M, x \rangle : M \text{ is a Turing machine and } M(x) \text{ halts in less than } |x|^2 \text{ steps}\right\}.$$
   (h) The set of true arithmetical formula is in NP.
   (i) There exists a recognizable set S for which no arithmetical formula exists with 1 free parameter, such that $n \in S$ if and only if $F(n)$ is true.
   (j) For some $n \in \mathbb{Z}$, let $\langle n \rangle$ be n in binary. Let
   $$\textsc{Composite} = \{\langle k\ell \rangle : k, \ell \in \mathbb{Z}, k \geq 2 \text{ and } \ell \geq 2\}$$
   The following algorithm shows that $\textsc{Composite}$ is in the complexity class P: On input M, check for all $k = 2, \ldots, M - 1$ whether k divides M. If a divisor is found: accept, otherwise reject.
   (k) The Halting problem H_{TM} is NP-hard.

2. The following problem is also called the *generalized Collatz* problem. Let (a, b, r_0) be a triple where $a = [a_1, \ldots, a_k]$ and $b = [b_1, \ldots, b_k]$ are lists of rational numbers of equal length and r_0 is a rational number. With each such triple we associate a computational process that operates on a single rational number r. Initially, $r = r_0$. In each step of the process, r is replaced by (3)
   $$r \cdot a_{\lfloor r \rfloor \bmod k} + b_{\lfloor r \rfloor \bmod k}.$$
   The computation halts if $r = 1$. Show that the set of triples for which the process halts is undecidable.

3. In the following solitaire game, you are given an $m \times m$ board. On each of its m^2 positions lies either a green stone, a red stone, or nothing at all. You play by removing stones from the board. You win if each column contains only stones of a single colour and each row contains at least one stone. Winning may or may not be possible depending upon the initial configuration. Prove that the problem 3SAT reduces to the set of winnable game configurations (and hence this set is NP-complete). (3)

4. Suppose that a language L is decided by a Turing machine M in time $o(n \log n)$. Show that for large $x \in L$ there exist two equal crossing sequences. (3)