Cautious Belief and Iterated Admissibility*

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March 2018

Abstract. We introduce novel notions of Cautiousness and Cautious Belief for Lexicographic Probability Systems, whose preference-based foundations build on a weak "infinitely more likely" relation between uncertain events. With this, we show that Iterated Admissibility characterizes the behavioral implications of Cautious Rationality and Common Cautious Belief of Cautious Rationality in a (continuous and belief-complete) terminal type structure. This contrasts with the negative result of Brandenburger, Friedenberg and Keisler (Econometrica 2008), according to which Rationality and Common Assumption of Rationality is impossible in all continuous and belief-complete type structures. On the other hand, we show that continuity and belief-completeness cannot substitute terminality for our positive result.

Keywords: Iterated Admissibility, Weak dominance, Infinitely More Likely, Lexicographic Probability Systems.

1 Introduction

Iterated Admissibility (henceforth IA), i.e., the iterated deletion of weakly dominated strategies, is an important and widely applied solution concept for games in strategic form.[1] Shimoji (2004) proved that, in dynamic games with generic payoffs at terminal nodes, IA coincides with Pearce's (1984) extensive-form rationalizability, a prominent

*A previous version of this paper has been circulated under the title "Common Assumption of Cautious Rationality and Iterated Admissibility." We are indebted to Pierpaolo Battigalli and Amanda Friedenberg for important inputs and suggestions about our work. We are grateful to George J. Mailath and two anonymous referees for their comments which have considerably improved the paper. We also thank Gabriele Beneduci, Adam Brandenburger, Martin Dufwenberg, Edward Green, Byung Soo Lee, Burkhard Schipper, Marciano Siniscalchi, Elias Tsakas and the attendants of our talks at AMES 2013 conference, LOFT 2014, Stern School of Business, Bocconi University, Scuola Normale Superiore and Politecnico di Milano for their valuable comments. The study has been funded within the framework of the Basic Research Program at the National Research University Higher School of Economics (HSE) and by the Russian Academic Excellence Project '5-100'. Financial support from European Research Council (STRATEMOTIONS–GA 324219) is gratefully acknowledged.

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[1] For instance, IA has been applied in voting (Moulin, 1984) and money-burning games (Ben Porath and Dekel, 1992).

solution concept whose foundations are well understood (see Battigalli and Siniscalchi 2002). Yet, while IA has an independent intuitive appeal, its theoretical foundations have proved to be elusive (see, e.g., Samuelson 1992). Thus, the decision-theoretic principles and the hypotheses about strategic reasoning that yield IA require careful scrutiny.

A recent literature—starting with the seminal contribution of Brandenburger, Friedenberg and Keisler (2008, henceforth BFK)—has tackled this issue building on two key ideas. The decision-theoretic aspects of the problem have been solved through the *Lexicographic Expected Utility* theory of Blume et al. (1991a). Lexicographic Expected Utility preferences are represented by *Lexicographic Probability Systems* (henceforth LPS's), i.e., lists of probabilistic conjectures in a priority order, each of which becomes relevant when the previous ones fail to identify a unique best alternative. In games with complete information, opponents' strategies constitute the only payoff-relevant uncertainty. In order to come up with an educated conjecture about opponents' strategies, a player naturally starts reasoning about opponents' beliefs and choice criteria. BFK modeled this aspect with the tools of *Epistemic Game Theory*, the formal, mathematical analysis of how players reason about each other in games.[2]

Inspired by BFK, we adopt Lexicographic Expected Utility and Epistemic Game Theory for our epistemic foundation of IA in finite games. However, we start from partially different basic principles. Specifically, we provide novel notions of *Rationality*, *Cautiousness* and *Cautious Belief* that, appropriately combined, justify the choice of iteratively admissible strategies in the following sense: IA characterizes the behavioral implications of Cautious Rationality and Common Cautious Belief of Cautious Rationality (henceforth $\mathrm{R^cCB^cR^c}$). Here we briefly introduce the main features of our approach.

1. We model players' beliefs as LPS's. In line with recent findings and developments in the field, we do not require the LPS's to be *mutually singular*, that is, we do not require the measures in the LPS to have (essentially) disjoint supports.

2. We define a simple notion of Cautiousness: every payoff-relevant event is deemed possible by the player. Together with lexicographic expected utility maximization (what we call "rationality"), Cautiousness justifies the choice of admissible strategies.

3. We use a monotone notion of "infinitely more likely" with the following simple interpretation: A player deems an event infinitely more likely than another if she prefers to bet on the first rather than on the second regardless of the winning prizes for the two bets.

4. We define a notion of Cautious Belief with the following preference-based foundation: Each payoff-relevant component of the event is deemed infinitely more likely than the complement of the event.

5. We show that in all sufficiently rich lexicographic type structures,[3] there are states consistent with $\mathrm{R^cCB^cR^c}$, and the behavioral implication of these epistemic con-

[2] See Dekel and Siniscalchi (2015) for a recent survey.

[3] Lexicographic type structures—i.e., type structures where each type's belief over opponents' strategies and types is an LPS—have been introduced by BFK as the analogue of standard type structures—i.e., type structures where beliefs are probability measures. Type structures are a convenient modelling device,

ditions is that players choose within the (non-empty) set of iteratively admissible strategies.

In the rest of the Introduction, we first discuss BFK and the related literature, then we illustrate our contribution, and finally we show by example some key differences between the two approaches.

1.1 Literature review

BFK justify each step of IA through novel notions of *Rationality* and *Assumption* of an event. Such notions are represented by events in a *lexicographic type structure*. More formally, for each player $i \in I$, there is a set T_i of types; each element $t_i \in T_i$ is associated with an LPS $\overline{\mu}_i = (\mu_i^1, ..., \mu_i^n)$ on strategies and types of the other players, viz. $S_{-i} \times T_{-i}$. A strategy-type pair (s_i, t_i) is said to be rational if, given the LPS $\overline{\mu}_i = (\mu_i^1, ..., \mu_i^n)$ associated with t_i,

(R1) s_i maximizes lexicographic expected utility under $\overline{\mu}_i$, and

(R2, "full-support") every open set $O \subseteq S_{-i} \times T_{-i}$ is not *Savage-null*, i.e., $\mu_i^l(O) > 0$ for some $l \leq n$.

Type t_i assumes an event $E \subseteq S_{-i} \times T_{-i}$ if

(A1) there is $m \leq n$ such that $\mu_i^l(E) = 1$ for $l \leq m$ and $\mu_i^l(E) = 0$ for $l > m$.

(A2) every *part* of E, i.e., every relatively open subset of E is not Savage-null.

The notion of Assumption captures the idea that every part of E is deemed "infinitely more likely" than not-E, according to the preference-based notion of Infinitely More Likely put forward by Blume et al. (1991a) (for a formal definition, see Definition A.6 in Appendix A).

BFK show that in every lexicographic type structure satisfying a richness condition, called *belief-completeness* (see Section 3.2 below for a formal definition), Rationality justifies Admissibility, and Rationality and m-th order mutual Assumption of Rationality justifies $m+1$ steps of the IA procedure. BFK also introduce the concept of Self-Admissible Set (henceforth, SAS), a weak-dominance analogue of best-reply set (Pearce 1984). BFK show that SAS's characterize the behavioral implications of Rationality and Common Assumption of Rationality across all lexicographic type structures, and that, in turn, every SAS can be epistemically justified by Rationality and Common Assumption of Rationality in some lexicographic type structure.

Three controversial features of BFK's analysis have fuelled a recent literature.

Issue 1. BFK restrict attention to mutually singular LPS's. There are reasons to find this restriction uncompelling. As noted by Lee (2013), mutual singularity is merely cosmetic when the space of uncertainty contains duplicates in terms of represented preferences, and this is typically the case for the "rich" type structures used for the analysis of IA.

due to Harsanyi (1967), to describe players' *hierarchies of beliefs*; that is, their beliefs about the play of the game (first-order beliefs), their beliefs about players' beliefs about play (second-order beliefs), and so on. See Section 3.2 for formal definitions.

Issue 2. R2 and A2 depend on the topology of the type spaces and do not have a clear interpretation in terms of belief hierarchies. We will show by example (Section 3.2) that a full-support and a non full-support type can represent the same hierarchy of beliefs. Analogously, Keisler and Lee (2015) show that a type can represent the same hierarchy of beliefs but assume different sets of events depending on the topology on the set of opponent's types.

Issue 3. BFK considered the natural conjecture that, in all belief-complete type structures with *continuous belief maps* (that is, the maps that associate each type with the corresponding LPS's are continuous), the strategies that survive all rounds of the IA procedure are exactly the strategies played at states with Rationality and Common Assumption of Rationality. However, they obtain a negative result: In every continuous, belief-complete type structure, Rationality and Common Assumption of Rationality is not possible at any state.

Dekel et al. (2016) provide a characterization of BFK's preference-based notion of Assumption that applies to all (not necessarily mutually singular) LPS's, by amending condition A1. With this, they prove that the results of BFK hold through in absence of mutual singularity.[4] Keisler and Lee (2015) construct a *discontinuous*, belief-complete type structure where Rationality and Common Assumption of Rationality is possible. They also show that such type structure generates the same set of hierarchies of beliefs as a continuous one. An immediate implication of this findings is that BFK's results hinge on topological details of the type structure that cannot be expressed in terms of belief hierarchies. Yang (2015) and Catonini (2013) obtain a non-empty "common assumption of rationality" event in two different continuous and belief-complete type structures. Their results are obtained by weakening condition A2 in the definition of Assumption as follows: only the intersections of the event with strategy-based cylinders and not with all open sets are required to be not Savage-null. Accordingly, Catonini also weakens BFK's notion of rationality with Cautiousness in place of full-support, and maintains mutual singularity of LPS's in the type structure.[5] On the contrary, Yang maintains BFK's notion of rationality and drops mutual singularity for the LPS's in the type structure,[6] but his definition of "Weak Assumption" and its preference-based foundation still crucially require mutual singularity. Lee (2016a) relaxes the traditional coherency condition on belief hierarchies while maintaining coherency of the represented preferences. With this, he identifies hierarchies of lexicographic beliefs without an upper bound on the length of the LPS's which cannot be induced by a type structure and capture Rationality and Common Assumption of Rationality, where Rationality is weakened with

[4]Taking the opposite direction, Heifetz et al. (2018) impose mutual singularity of the first-order beliefs. With this, they depart from IA and derive a new solution concept called "Comprehensive Rationalizability." Their approach is akin to ours in the interpretation and representation of players' caution.

[5]More precisely, Catonini (2013) constructs the "*canonical*" type structure that captures all lexicographic hierarchies of beliefs that satisfy *coherency*, "mutual singularity" (of the LPS's on opponents' strategies and hierarchies that summarizes the hierarchy), and common certain belief thereof. A more detailed construction of this type structure and a thorough study of its properties is carried on in Catonini and De Vito (2016)

[6]In particular, Yang (2015) constructs the canonical type structure that captures all the lexicographic hierarchies of beliefs that satisfy coherency and common (certain) belief in coherence.

Cautiousness in place of full-support.[7] We find Lee's approach extremely interesting, thus we show in a companion paper (Catonini and De Vito 2017) that our results can be replicated in the larger hierarchical space he considers.

1.2 Our contribution

All the papers mentioned above target the controversies in BFK only partially, often leaving new unanswered questions. We note that the origin of the controversies lies in the chosen notion of "infinitely more likely." Blume et al. (1991a) introduce a notion of "infinitely more likely" in the context of a finite space of uncertainty without Savage-null events. Further, they restrict attention to mutually singular LPS's to obtain the following two desirable properties: (i) the support of each measure in the LPS is infinitely more likely than the support of each subsequent measure; (ii) if all non-empty subsets of E are infinitely more likely than F, then E is infinitely more likely than F. To extend these properties to infinite spaces, BFK focus on full-support, mutually singular LPS's, and introduce condition A2 of Assumption (cf. Appendix A, final comment). Thus, Assumption requires that every part of E is infinitely more likely than not-E. As noted by Keisler and Lee (2015), this represents an elusive degree of "caution" towards the assumed event, whose consequences change dramatically according to the topology on the set of types.

Therefore, we tackle the problem at the root and we rebuild the foundations of IA with different notions of "infinitely more likely" and caution. The simple notion of "infinitely more likely" we adopt, originally introduced by Lo (1999), has properties (i) and (ii) also in absence of mutual singularity and in presence of Savage-null events. Moreover, it has other intuitive properties, foremost *monotonicity*: If event E is deemed infinitely more likely to occur than event F, then every event G satisfying $E \subseteq G$ is also deemed infinitely more likely to occur than F. *Cautiousness* represents a clear condition on the belief hierarchy, namely full-support of first-order beliefs. When a player focuses on an event that she deems infinitely more likely than its complement, the same cautious attitude yields *Cautious Belief* of the event. Then, an event E is cautiously believed if and only if all its payoff-relevant components are deemed infinitely more likely than not-E.[8] Differently from BFK, the cautious attitude towards the event is driven only by the primitive, payoff-relevant uncertainty.

Equipped with these notions, we can construct in a simple way our $\mathrm{R^cCB^cR^c}$ event in every *terminal* type structure, that is, a type structure that "contains" all hierarchies of beliefs (Theorem 1). We also show that the notion of belief-completeness is not, in general, a sufficient "richness" condition for a type structure: There exist (continuous) belief-complete type structures where $\mathrm{R^cCB^cR^c}$ is impossible. This is so because a belief-complete type structure is not necessarily terminal—see Theorem 3.

[7] Differently from the full-support condition of BFK, Cautiousness can be expressed in terms of belief hierarchies (full support of first-order beliefs) and is therefore suited for Lee's epistemic analysis in the formalism of hierarchies instead of type structures.

[8] Note that both Assumption and Cautious Belief have a preference-based foundation of the following kind: A suitable family of Borel subsets of E is, in a suitable sense, infinitely more likely than not-E. This is true also for Weak Assumption with the same notion of infinitely more likely as for Assumption and the same family of Borel subsets as for Cautious Belief *if the preferences can be represented by a mutually singular LPS*, but not otherwise.

Our approach has two further advantages: First, it clarifies an important aspect of the foundations of SAS (and of IA); second, it isolates the epistemic hypothesis that distinguishes IA from Permissibility (Brandenburger 1992; see also Borgers 1994 and Ben-Porath 1997), i.e., the Dekel-Fudenberg procedure (Dekel and Fudenberg 1990).

We show that SAS's characterizes the behavioral implications of $\mathrm{R}^{\mathrm{c}}\mathrm{CB}^{\mathrm{c}}\mathrm{R}^{\mathrm{c}}$ across all type structures, and that, in turn, every SAS can be justified by $\mathrm{R}^{\mathrm{c}}\mathrm{CB}^{\mathrm{c}}\mathrm{R}^{\mathrm{c}}$ in some *cautious type structure*, that is, a type structure where all types are cautious (Theorem 2). In BFK, instead, the type structure may need types without full-support of the associated LPS. This induced BFK to leave the following open question (open question C, p. 333): Which SAS's can be justified within type structures with only full-support types? With Cautious Belief in place of Assumption, all.[9] Differently from BFK, we can do away with incautious types also for the justification of IA. This allows us to provide an alternate epistemic characterization of IA under *transparency of Cautiousness* (see Section 5 for details).

The Dekel-Fudenberg procedure consists of one round of elimination of all weakly dominated strategies, followed by iterated elimination of strictly dominated strategies. In Catonini and De Vito (2018), we say that an event E is "weakly believed" if E is infinitely more likely than not-E, where the notion of "infinitely more likely" is in the sense of Lo. With this, we show that Permissibility characterizes the behavioral implications of Cautious Rationality and Common Weak Belief of Cautious Rationality. Thus, the difference between Weak Belief and Cautious Belief, hence between IA and Permissibility, lies in the presence or not of cautiousness towards the believed events.

Finally, it should be pointed out that all the results in this paper do not hinge on the topology of type spaces. Therefore, we provide a justification of IA using *expressible* epistemic assumptions about rationality and beliefs, that is, assumptions which are expressible in a language describing primitive terms (strategies) and terms derived from the primitives (beliefs about strategies, beliefs about strategies and beliefs of others, etc.)—cf. Battigalli et al. (2011).

1.3 Infinitely More Likely, Cautious Belief and Assumption: an example

Consider the following situation. There are two players, Ann (a) and Bob (b). Bob has two strategies, u and d, and can be of two types, t_b' and t_b''. Suppose that u strictly dominates d. Ann thinks that Bob is of type t_b' and rational (which from now on will simply mean "lexicographic expected utility maximizer"). However, she also entertains the hypothesis that Bob might be of type t_b''. In this case, she thinks that Bob chooses at random between u and d. Ann's unique type t_a is associated with the LPS $\overline{\mu}_a = (\mu_a^1, \mu_a^2)$, which is summarized in the following tables.

μ_a^1	t_b'	t_b''
u	1	0
d	0	0

μ_a^2	t_b'	t_b''
u	0	1/2
d	0	1/2

[9] Our epistemic conditions only require Cautiousness in place of full-support, but this is not crucial: see the proof of Theorem 2 in Appendix B.

In her primary theory (μ_a^1), Ann assigns probability 1 to Bob's rationality, i.e., to the event $R_b = \{(u, t_b'), (u, t_b'')\}$. So, she considers Bob's rationality infinitely more likely (*à la* Lo) than Bob's irrationality. Since Bob's rationality has only one possible payoff-relevant implication (u), this also means that Ann cautiously believes in Bob's rationality.

However, Ann does not (weakly) assume Bob's rationality. It is true that she deems $\{(u, t_b')\}$ infinitely more likely (*à la* Blume et al. 1991a) than irrationality. But this is not true for the larger event R_b (indeed, the "infinitely more likely" relation of Blume at al, 1991a, is *not* monotonic). This is because Ann deems $\{(u, t_b'')\}$ "possibile" but not infinitely more likely than not-R_b: Her secondary theory (μ_a^2) gives positive probability both to the event that type t_b'' plays the dominated action, and to the event that t_b'' plays the dominant action, albeit by chance.[10] Paradoxically, if we modify μ_a^2 by shifting all the probability from the rational pair (u, t_b'') to the irrational pair (d, t_b''), Ann would now deem Bob's rationality infinitely more likely than irrationality. By contrast, Infinitely More Likely *à la* Lo (and consequently Cautious Belief) has the natural property of being preserved after an increase of the probability assigned to all Borel subsets of the infinitely more likely event, under any component-measure of the LPS.

Consider now the following situation, which is analyzed formally in Appendix A (Example A.1).

μ_a^1	t_b'	t_b''
u	1/2	1/2
d	0	0

μ_a^2	t_b'	t_b''
u	0	1/2
d	0	1/2

Ann still cautiously believes in Bob's rationality, and now one could expect that she considers Bob's rationality infinitely more likely (*à la* Blume et al. 1991a) than irrationality as well. Indeed, both rational strategy-type pairs of Bob, (u, t_b') and (u, t_b''), are deemed infinitely more likely than irrationality, and the event R_b coincides with the support of μ_a^1. Still, Ann does not consider Bob's rationality infinitely more likely than Bob's irrationality. Note that $\overline{\mu}_a$ is not mutually singular. In Appendix A, we provide the formal definition of infinitely more likely *à la* Blume et al. (1991a) and we show why it is violated in this example for R_b and its complement.

To summarize: If E is infinitely more likely than F (in the sense of Blume et al. 1991a), event G is Savage-null, and event H is not Savage-null, then $E \cup G$ is infinitely more likely than F, while $E \cup H$ may not, even when H is infinitely more likely than F too (in absence of mutual singularity).

1.4 Structure of the paper

The remainder of the paper is structured as follows. Section 2 introduces IA and SAS. Section 3 provides formal definitions of LPS's and type structures. Section 4 presents our notions of Cautiousness and Cautious Belief. In Section 5 we carry on the epistemic analysis of IA and SAS's. Appendix A introduces the language for the preference-based treatment of Cautious Belief, and compares formally the notion of Infinitely More Likely

[10] The fact that type t_b'' randomly chooses the dominant strategy is of course only one possible interpretation. However, it seems plausible to allow for it in a theory that entertains irrational play.

due to Lo (1999) and the one due to Blume et al. (1991a). Appendix B and Appendix C collect the proofs omitted from the main text. Appendix D contains technical results pertaining to the measurability of the relevant events.

2 Iterated Admissibility and Self-Admissible Sets

Throughout the paper, we fix a finite game $G = \langle I, (S_i, u_i)_{i\in I}\rangle$, where I is a two-players set and, for every $i \in I$, S_i is the set of strategies with $|S_i| \geq 2$ and $u_i : S_i \times S_{-i} \to \mathbb{R}$ is the payoff function.[11] Each strategy set S_i is given the obvious topology, i.e., the discrete topology. We let $\mathcal{M}(X)$ denote the set of all Borel probability measures on a topological space X. So, for every $i \in I$, we define the expected payoff function π_i by extending u_i on $\mathcal{M}(S_i) \times \mathcal{M}(S_{-i})$ in the usual way:

$$\pi_i(\sigma_i, \sigma_{-i}) = \sum_{(s_i,s_{-i})\in S_i\times S_{-i}} \sigma_i(s_i)\sigma_{-i}(s_{-i})u_i(s_i, s_{-i}).$$

The notion of admissible strategy is standard.

Definition 1 *Fix a set $X_i \times X_{-i} \subseteq S_i \times S_{-i}$. A strategy $s_i \in S_i$ is* ***admissible with respect to*** *$X_i \times X_{-i}$ if and only if there exists $\sigma_{-i} \in \mathcal{M}(S_{-i})$ with $\sigma_{-i}(X_{-i}) = 1$ such that $\sigma_{-i}(s_{-i}) > 0$ for all $s_{-i} \in X_{-i}$, and $\pi_i(s_i, \sigma_{-i}) \geq \pi_i(s'_i, \sigma_{-i})$ for every $s'_i \in X_i$. If strategy $s_i \in S_i$ is admissible with respect to $S_i \times S_{-i}$, we simply say that s_i is admissible.*

Remark 1 *Fix a set $X_i \times X_{-i} \subseteq S_i \times S_{-i}$. A strategy $s_i \in S_i$ is* ***weakly dominated with respect to*** *$X_i \times X_{-i}$ if there exists $\sigma_i \in \mathcal{M}(S_i)$ with $\sigma_i(X_i) = 1$ such that $\pi_i(\sigma_i, s_{-i}) \geq \pi_i(s_i, s_{-i})$ for every $s_{-i} \in X_{-i}$ and $\pi_i(\sigma_i, s'_{-i}) > \pi_i(s_i, s'_{-i})$ for some $s'_{-i} \in X_{-i}$. A standard result (Pearce 1983, Lemma 4) states that a strategy $s_i \in S_i$ is not weakly dominated with respect to $X_i \times X_{-i}$ if and only if it is admissible with respect to $X_i \times X_{-i}$.*

The set of iteratively admissible strategies (henceforth IA set) is defined inductively.

Definition 2 *For each $i \in I$, let $S_i^0 = S_i$, and for every $m \in \mathbb{N}$, let S_i^m be the set of all $s_i \in S_i^{m-1}$ that are admissible with respect to $S_i^{m-1} \times S_{-i}^{m-1}$. A strategy $s_i \in S_i^m$ is called m-**admissible**. A strategy $s_i \in S_i^\infty = \cap_{m=0}^\infty S_i^m$ is called* ***iteratively admissible****.*

By finiteness of the game, it follows that $S_i^m \neq \emptyset$ for all $m \in \mathbb{N}$, and, since $S_i^m \supseteq S_i^{m+1}$ for all $m \in \mathbb{N}$, that there exists $M \in \mathbb{N}$ such that $S_i^\infty = S_i^M$. Consequently, the IA set $S_i^\infty \times S_{-i}^\infty$ is non-empty.

To formally introduce SAS's, we need an additional definition. Say that a strategy $s'_i \in S_i$ **supports** $s_i \in S_i$, if there exists a mixed strategy $\sigma_i \in \mathcal{M}(S_i)$ such that $\sigma_i(s'_i) > 0$ and $\pi_i(\sigma_i, s_{-i}) = \pi_i(s_i, s_{-i})$ for all $s_{-i} \in S_{-i}$.

[11] The assumption that there are two players is only for simplicity of exposition. The analysis can be trivially extended to more than two players.

Definition 3 *A set $Q_i \times Q_{-i} \subseteq S_i \times S_{-i}$ is a **Self-Admissible Set (SAS)** if, for every player i,*

(a) each $s_i \in Q_i$ is admissible,

(b) each $s_i \in Q_i$ is admissible with respect to $S_i \times Q_{-i}$,

(c) for every $s_i \in Q_i$, if s_i' supports s_i then $s_i' \in Q_i$.

Every finite game admits an SAS—in particular, the IA set is an SAS. But, as shown by BFK, many games possess other SAS's, which can be even disjoint from the IA set. A comprehensive analysis of the properties of SAS's in a wide class of games is given by Brandenburger and Friedenberg (2010).

3 Lexicographic beliefs and lexicographic type structures

3.1 Lexicographic probability systems

We start with some basic standard assumptions. All the sets considered in this paper are assumed to be Polish spaces (that is, topological spaces that are homeomorphic to complete, separable metrizable spaces), and they are endowed with the Borel σ-field. We let Σ_X denote the Borel σ-field of a Polish space X, the elements of which are called *events*. When it is clear from the context, we suppress reference to Σ_X and simply write X to denote a measurable space.

Given a countable collection $(X_n)_{n\in\mathbb{N}}$ of pairwise disjoint Polish spaces, the set $X = \cup_{n\in\mathbb{N}} X_n$ is endowed with the *direct sum topology*,[12] so that X is a Polish space. Moreover, we endow each finite or countable product of Polish spaces with the product topology, hence the product space is Polish as well.

Recall that $\mathcal{M}(X)$ denotes the set of Borel probability measures on a topological space X. The set $\mathcal{M}(X)$ is endowed with the *weak**-topology. So, if X is Polish, then $\mathcal{M}(X)$ is also Polish. We denote by $\mathcal{N}(X)$ (resp. $\mathcal{N}_n(X)$) the set of all finite (resp. length-n) sequences of Borel probability measures on X, that is,

$$\begin{aligned}\mathcal{N}(X) &= \cup_{n\in\mathbb{N}} \mathcal{N}_n(X) \\ &= \cup_{n\in\mathbb{N}} (\mathcal{M}(X))^n .\end{aligned}$$

Each $\overline{\mu} = (\mu^1, ..., \mu^n) \in \mathcal{N}(X)$ is called **lexicographic probability system** (LPS). In view of our assumptions, the topological space $\mathcal{N}(X)$ is Polish.

For every Borel probability measure μ on X, the support of μ, denoted by $\mathrm{Supp}\mu$, is the smallest closed subset $C \subseteq X$ such that $\mu(C) = 1$. The support of an LPS $\overline{\mu} = (\mu^1, ..., \mu^n) \in \mathcal{N}(X)$ is thus defined as $\mathrm{Supp}\overline{\mu} = \cup_{l\leq n} \mathrm{Supp}\mu^l$. So, an LPS $\overline{\mu} = (\mu^1, ..., \mu^n) \in \mathcal{N}(X)$ is of **full-support** if $\cup_{l\leq n}\mathrm{Supp}\mu^l = X$. We write $\mathcal{N}^+(X)$ for the set of full-support LPS's.

[12] In this topology, a set $O \subseteq X$ is open if and only if $O \cap X_n$ is open in X_n for all $n \in \mathbb{N}$. The assumption that the spaces X_n are pairwise disjoint is without any loss of generality, since they can be replaced by a homeomorphic copy, if needed (see Engelking 1989, p. 75).

Suppose we are given Polish spaces X and Y, and a Borel map $f : X \to Y$. The map $\widetilde{f} : \mathcal{M}(X) \to \mathcal{M}(Y)$, defined by

$$\widetilde{f}(\mu)(E) = \mu\left(f^{-1}(E)\right), \quad E \in \Sigma_Y,\, \mu \in \mathcal{M}(X),$$

is called the image (or pushforward) measure map of f. For each $n \in \mathbb{N}$, the map $\widehat{f}_{(n)} : \mathcal{N}_n(X) \to \mathcal{N}_n(Y)$ is defined by

$$\left(\mu^1, ..., \mu^n\right) \mapsto \widehat{f}_{(n)}\left(\left(\mu^1, ..., \mu^n\right)\right) = \left(\widetilde{f}\left(\mu^k\right)\right)_{k \leq n}.$$

Thus the map $\widehat{f} : \mathcal{N}(X) \to \mathcal{N}(Y)$ defined by

$$\widehat{f}(\overline{\mu}) = \widehat{f}_{(n)}(\overline{\mu}),\, \overline{\mu} \in \mathcal{N}_n(X),$$

is called the **image LPS map of** f. Alternatively put, the map $\widehat{f}$ is the *union* of the maps $\left(\widehat{f}_{(n)}\right)_{n \in \mathbb{N}}$, and it is Borel measurable (see Fact D.1 in Appendix D).

Given Polish spaces X and Y, we let Proj_X denote the canonical projection from $X \times Y$ onto X; in view of our assumption, the map Proj_X is continuous. The marginal measure of $\mu \in \mathcal{M}(X \times Y)$ on X is defined by $\mathrm{marg}_X \mu = \widetilde{\mathrm{Proj}}_X(\mu)$. Consequently, the marginal of $\overline{\mu} \in \mathcal{N}(X \times Y)$ on X is defined by $\overline{\mathrm{marg}}_X \overline{\mu} = \widehat{\mathrm{Proj}}_X(\overline{\mu})$, and the function $\widehat{\mathrm{Proj}}_X : \mathcal{N}(X \times Y) \to \mathcal{N}(X)$ is continuous and surjective (see Fact D.2 in Appendix D).

3.2 Lexicographic type structures

A *type structure* formalizes Harsanyi's implicit approach to model hierarchies of beliefs. The following is a natural generalization of the standard definition of epistemic type structure with beliefs represented by probability measures (i.e., length-1 LPS's; cf. Heifetz and Samet 1998).

Definition 4 *An $(S_i)_{i \in I}$-**based lexicographic type structure** (henceforth, a "type structure") is a structure $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i \in I}$ where*

1. *for each $i \in I$, T_i is a Polish space;*
2. *for each $i \in I$, the function $\beta_i : T_i \to \mathcal{N}(S_{-i} \times T_{-i})$ is Borel measurable.*

*We call each space T_i **type space** and we call each β_i **belief map**.*[13] *Members of type spaces, viz. $t_i \in T_i$, are called **types**. Each element $(s_i, t_i)_{i \in I} \in \prod_{i \in I} S_i \times T_i$ is called **state (of the world)**.*

[13] Some authors (e.g., Battigalli and Siniscalchi 1999, Heifetz and Samet 1998) use the terminology "type space" for what is called "type structure" here.

Type structures generate a collection of hierarchies of beliefs for each player. For instance, type t_i's first-order belief is an LPS on S_{-i}, and is given by $\overline{\mathrm{marg}}_{S_{-i}}\beta_i(t_i)$. A standard inductive procedure (see Catonini and De Vito, 2016, for details) shows how to provide an explicit description of a hierarchy induced by a type.

We will be interested in type structures with one or more of the following features, which do not make reference to hierarchies of beliefs or other type structures.

Definition 5 *A type structure $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i\in I}$ is*

- ***finite*** *if the cardinality of each type space T_i is finite;*
- ***compact*** *if each type space T_i is compact;*
- ***belief-complete*** *if each belief map β_i is onto;*
- ***continuous*** *if each belief map β_i is continuous.*

The idea of belief-completeness was introduced by Brandenburger (2003) and adapted to the present context by BFK.[14] Note that each type space in a belief-complete type structure has the cardinality of the continuum. Finite type structures are trivially compact and continuous, but not belief-complete. No belief-complete lexicographic type structure is also compact and continuous. To see this, observe that if the type structure is compact and continuous, each $\beta_i(T_i)$ is compact but the space $\mathcal{N}(S_{-i} \times T_{-i})$ is not compact,[15] hence β_i cannot be onto.

We now introduce the notion of type morphism, which captures the idea that a type structure $\mathcal{T}$ is "contained in" another type structure $\mathcal{T}^*$. In what follows, given a type structure $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i\in I}$, we let T denote the Cartesian product of type spaces, that is, $T = \Pi_{i\in I} T_i$. Moreover, for a space X, we let Id_X denote the identity map on X, that is, $\mathrm{Id}_X(x) = x$ for all $x \in X$.

Definition 6 *Let $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i\in I}$ and $\mathcal{T}^* = \langle S_i, T_i^*, \beta_i^* \rangle_{i\in I}$ be two type structures. For each $i \in I$, let $\varphi_i : T_i \to T_i^*$ be a measurable map such that*

$$\beta_i^* \circ \varphi_i = \left(\widehat{\mathrm{Id}_{S_{-i}}, \varphi_{-i}}\right) \circ \beta_i.$$

*Then the function $(\varphi_i)_{i\in I} : T \to T^*$ is called **type morphism (from $\mathcal{T}$ to $\mathcal{T}^*$)**.*

*The morphism is called **bimeasurable** if the map $(\varphi_i)_{i\in I}$ is Borel bimeasurable. The morphism is called **type isomorphism** if the map $(\varphi_i)_{i\in I}$ is a Borel isomorphism. Say $\mathcal{T}$ and $\mathcal{T}^*$ are **isomorphic** if there is a type isomorphism between them.*

[14] BFK (Proposition 7.2) show with a simple, elegant argument the existence of a belief-complete lexicographic type structure.

[15] The space $\mathcal{M}(X)$ is compact if and only if X is compact, and this in turn implies that the space $\mathcal{N}_n(X)$ is also compact for every *finite* $n \in \mathbb{N}$. But the same conclusion does not hold for the space $\mathcal{N}(X)$. This is an instance of a well-known mathematical fact (see Theorem 2.2.3 in Engelking 1989): If $(X_\theta)_{\theta\in\Theta}$ is an indexed family of non-empty compact spaces with $|X_\theta| > 1$ for all $\theta \in \Theta$, then the direct sum $\cup_{\theta\in\Theta} X_\theta$ is compact if and only if the right-directed set Θ is finite.

A type morphism requires consistency between the function $\varphi_i : T_i \to T_i^*$ and the induced function $\left(\widehat{\mathrm{Id}_{S_{-i}}, \varphi_{-i}}\right) : \mathcal{N}\left(S_{-i} \times T_{-i}\right) \to \mathcal{N}\left(S_{-i} \times T_{-i}^*\right)$. That is, the following diagram commutes:

$$\begin{array}{ccc} T_i & \xrightarrow{\beta_i} & \mathcal{N}(S_{-i} \times \ T_{-i}) \\ \Big\downarrow \varphi_i & & \Big\downarrow (\widehat{Id_{S_{-i}}, \varphi_{-i}}) \\ T_i^* & \xrightarrow{\beta_i^*} & \mathcal{N}(S_{-i} \times \ T_{-i}^*) \end{array}$$

Thus, a type morphism maps $\mathcal{T}$ into $\mathcal{T}^*$ in a way that preserves the beliefs associated with types.

The notion of type morphism does not make reference to hierarchies of beliefs. But, as one should expect, the important property of type morphisms is that they preserve the explicit description of lexicographic belief hierarchies: the $(S_i)_{i \in I}$-based belief hierarchy generated by a type $t_i \in T_i$ in $\mathcal{T}$ is also generated by its image $\varphi_i(t_i) \in T_i^*$ in $\mathcal{T}^*$.[16] Heifetz and Samet (1998, Proposition 5.1) provide this result for the case of standard type structures; the generalization to lexicographic type structures is straightforward (see Catonini and De Vito, 2016).

Next, we introduce the notion of terminality of a type structure.

Definition 7 *Fix a class* $\mathbb{T}$ *of type structures. A type structure* $\mathcal{T}^* = \langle S_i, T_i^*, \beta_i^* \rangle_{i \in I}$ *is* ***terminal with respect to*** $\mathbb{T}$ *if for every type structure* $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i \in I}$ *in* $\mathbb{T}$*, there is a type morphism from* $\mathcal{T}$ *to* $\mathcal{T}^*$*.*

Whenever $\mathcal{T}^*$ is terminal with respect to the class of all type structures, we simply say, as customary, that $\mathcal{T}^*$ is *terminal.* In Section 5 we will show that $\mathrm{R^cCB^cR^c}$ justifies IA in every type structure which is terminal with respect to the class of all finite type structures, and that such a type structure exists.

4 Cautiousness and Cautious Belief

For this section, append to the game G a type structure $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i \in I}$.

4.1 Rationality and Cautiousness

For any two vectors $x = (x_l)_{l=1}^n, y = (y_l)_{l=1}^n \in \mathbb{R}^n$, we write $x \geq_L y$ if either (1) $x_l = y_l$ for every $l \leq n$, or (2) there exists $m \leq n$ such that $x_m > y_m$ and $x_l = y_l$ for every $l < m$.

[16] Put differently, every type morphism is also a *hierarchy morphism*, i.e., a map between type structures which preserves the hierarchies of beliefs associated with types. See Friedenberg and Meier (2011) for a general analysis on the relationship between hierarchy and type morphisms.

Definition 8 *. A strategy $s_i \in S_i$ is **optimal under** $\beta_i(t_i) = (\mu_i^1, ..., \mu_i^n) \in \mathcal{N}(S_{-i} \times T_{-i})$ if*

$$\left(\pi_i(s_i, \mathrm{marg}_{S_{-i}} \mu_i^l)\right)_{l=1}^n \geq_L \left(\pi_i(s_i', \mathrm{marg}_{S_{-i}} \mu_i^l)\right)_{l=1}^n, \ \forall s_i' \in S_i.$$

*We say that s_i is a **lexicographic best reply to** $\overline{\mathrm{marg}}_{S_{-i}} \beta_i(t_i)$ on S_{-i} if it is optimal under $\beta_i(t_i)$.*

This is the usual definition of optimality for a strategy, but this time optimality is taken lexicographically. We now introduce the notion of Cautiousness.

Definition 9 *A type $t_i \in T_i$ is **cautious** if $\overline{\mathrm{marg}}_{S_{-i}} \beta_i(t_i) \in \mathcal{N}^+(S_{-i})$.*

In words, the notion of Cautiousness requires that the first-order belief of a type be a full-support LPS. It is therefore a condition that can be expressed in terms of primitives of the model (i.e., hierarchies of beliefs), and it requires that every payoff-relevant event, viz. $\{s_{-i}\} \times T_{-i}$, be assigned strictly positive probability by at least one of the component measures of the LPS $\beta_i(t_i)$.

For strategy-type pairs we define the following notions.

Definition 10 *Fix a strategy-type pair $(s_i, t_i) \in S_i \times T_i$.*

1. *Say (s_i, t_i) is **rational** if s_i is optimal under $\beta_i(t_i)$.*
2. *Say (s_i, t_i) is **cautiously rational** if it is rational and t_i is cautious.*

We let R_i^c denote the set of all cautiously rational strategy-type pairs of player i. As one should expect, Cautious Rationality guarantees Admissibility.

Proposition 1 *If strategy-type pair $(s_i, t_i) \in S_i \times T_i$ is cautiously rational, then s_i is admissible.*

Clearly, if a type $t_i \in T_i$ is associated with a full-support LPS, viz. $\beta_i(t_i) \in \mathcal{N}^+(S_{-i} \times T_{-i})$, then it is cautious, but the reverse implication does not hold. We will show that Cautiousness and Cautious Rationality are invariant to type morphisms (see Lemma 2). Full-support is instead a topological condition which is not preserved by type morphisms, not even when two type structures are isomorphic (so that they generate the same set of hierarchies of beliefs). The following example elaborates on this point.

Example 1 *Let $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i \in I}$ be a symmetric type structure in which $T_i = [0,1]$, and the set $[0,1]$ is endowed with the standard topology. We can construct an isomorphic type structure $\mathcal{T}^* = \langle S_i, T_i^*, \beta_i^* \rangle_{i \in I}$ as follows. Let $T_i^* = [0,1]$, but T_i^* is endowed with the Polish topology which makes the set $\{1\}$ clopen, and which generates the same Borel σ-field as the one generated by the standard topology (see Theorem 3.2.4 in Srivastava 1998). So, T_i and T_i^* are Borel isomorphic, but not homeomorphic. For each $i \in I$, let $\varphi_i : T_i \to T_i^*$ be the identity map. Moreover, each belief map β_i^* satisfies $\beta_i^* = \widehat{(\mathrm{Id}_{S_i}, \varphi_i)} \circ \beta_i \circ (\varphi_i)^{-1}$. It is easy to check that $(\varphi_i)_{i \in I}$ is a type isomorphism. Let $t_i \in T_i$ be such that the associated LPS $\beta_i(t_i)$ is a product probability measure on $S_{-i} \times T_{-i}$ as follows: $\beta_i(t_i) = \nu \times \lambda$, where ν is a full-support probability measure on S_{-i}, and λ is the Lebesgue measure. We clearly have that $\beta_i(t_i) \in \mathcal{N}_1^+(S_{-i} \times T_{-i})$ and $\beta_i(t_i)(S_{-i} \times \{1\}) = 0$. The set $S_{-i} \times \{1\}$ is closed in T_i, but (cl)open in T_i^*. It turns out that $\beta_i^*(\varphi_i(t_i))(S_{-i} \times \{1\}) = 0$, hence $\varphi_i(t_i) \in T_i^*$ is not associated with a full-support LPS, although it induces the same hierarchy of beliefs as $t_i \in T_i$.*

4.2 Infinitely More Likely and Cautious Belief

Following Lo (1999), we say that player i deems event E infinitely more likely than event F if she prefers to bet on E rather than on F no matter the prizes for the two bets. We formalize this preference-based notion in Appendix A, where we introduce the appropriate language.[17] Here, we provide the equivalent definition of "infinitely more likely" in terms of the LPS that represents player i's preferences. This equivalence is formally proved in Appendix A.

Given an LPS $\overline{\mu}_i = (\mu_i^1, ..., \mu_i^n) \in \mathcal{N}(S_{-i} \times T_{-i})$ and an event $E \subseteq S_{-i} \times T_{-i}$, let

$$\mathcal{I}_{\overline{\mu}_i}(E) = \inf \left\{ l \in \{1, ..., n\} \left| \mu_i^l(E) > 0 \right. \right\}.$$

Definition 11 *Fix two disjoint events $E, F \subseteq S_{-i} \times T_{-i}$. Say that E is* ***infinitely more likely*** *than F under $\overline{\mu}_i$ if $\mathcal{I}_{\overline{\mu}_i}(E) < \mathcal{I}_{\overline{\mu}_i}(F)$.*

It is straightforward to see that "infinitely more likely" is monotone.

Remark 2 *If E is infinitely more likely than F under $\overline{\mu}_i$ and G is an event such that $E \subseteq G$, then G is infinitely more likely than F under $\overline{\mu}_i$.*

Consider now the following two attitudes of player i towards an event E. First, player i deems E infinitely more likely than its complement. Second, player i has a cautious attitude towards the event: Before considering its complement, she considers all the possible payoff-relevant consequences of the event. The notion of Cautious Belief captures both attitudes.

[17] At the end of Appendix A, we also provide an alternate representation of the preference-based notion of "infinitely more likely" in terms of the "infinitely greater" relation between non-standard real probability values.

Definition 12 *Fix a non-empty event* $E \subseteq S_{-i} \times T_{-i}$ *and a type* $t_i \in T_i$ *with* $\beta_i(t_i) = (\mu_i^1, ..., \mu_i^n)$. *We say that* E *is* ***cautiously believed under*** $\beta_i(t_i)$ ***at level*** $m \leq n$ *if the following conditions hold:*

(i) $\mu_i^l(E) = 1$ *for all* $l \leq m$*;*

(ii) *for every elementary cylinder* $C = \{s_{-i}\} \times T_{-i}$*, if* $E \cap C \neq \emptyset$ *then* $\mu_i^l(E \cap C) > 0$ *for some* $l \leq m$*.*

We say that E *is* ***cautiously believed under*** $\beta_i(t_i)$ *if it is cautiously believed under* $\beta_i(t_i)$ *at some level* $m \leq n$*.*

We say that $t_i \in T_i$ ***cautiously believes*** E *if* E *is cautiously believed under* $\beta_i(t_i)$*.*

Condition (i) captures the first attitude. Under condition (i), condition (ii) is equivalent to saying that player i deems all *payoff-relevant parts* of E (i.e., the intersections of E with each strategy-based cylinder) infinitely more likely than not-E, so it captures the second attitude.

Differently from (Weak) Assumption, Definition 12 does not impose any requirement on the measures μ^l with $l > m$ of the LPS. This makes Cautious Belief weaker than (Weak) Assumption[18] and reflects the fact that the notion of "infinitely more likely" we adopt is weaker than the one put forward by Blume et al. (1991a)—see Appendix A.

The conceptual consistency between Cautiousness and Cautious Belief is highlighted by the following connection.

Remark 3 *A type* $t_i \in T_i$ *is cautious if and only if* t_i *cautiously believes* $S_{-i} \times T_{-i}$*.*

We now present some important properties of Cautious Belief.

Proposition 2 *Fix a type* $t_i \in T_i$ *with* $\beta_i(t_i) = (\mu_i^1, ..., \mu_i^n)$*.*

1. *Fix non-empty events* $E_1, E_2, ...$ *in* $S_{-i} \times T_{-i}$*. If, for each* k*, type* t_i *cautiously believes* E_k*, then* t_i *cautiously believes* $\cap_k E_k$ *and* $\cup_k E_k$*.*

2. *A non-empty event* $E \subseteq S_{-i} \times T_{-i}$ *is cautiously believed under* $\beta_i(t_i)$ *if and only if there exists* $m \leq n$ *such that* $\beta_i(t_i)$ *satisfies condition (i) of Definition 12 plus the following condition:*

   (ii') $\cup_{l \leq m} \mathrm{Suppmarg}_{S_{-i}} \mu_i^l = \mathrm{Proj}_{S_{-i}}(E)$*.*

[18] With respect to Assumption, but not to Weak Assumption, Condition (ii) of Definition 12 is also weaker than the corresponding Condition A2 of Assumption stated in the Introduction. Interestingly, Dekel et al. (2016) put forward also the notion of "topological weak-dominance assumption," which shares A2 with Assumption and Condition (i) with Cautious Belief. Indeed, "topological weak-dominance assumption" can be given a preference-based foundation with "infinitely more likely" *à la* Lo as well.

Proposition 2.1 states that Cautious Belief satisfies one direction of conjunction as well as one direction of disjunction. Proposition 2.2 can be viewed as a "marginalization" property of Cautious Belief: If E is cautiously believed under $\beta_i(t_i)$, then (a) $\mathrm{Proj}_{S_{-i}}(E)$ is infinitely more likely than $S_{-i}\backslash\mathrm{Proj}_{S_{-i}}(E)$ under $\overline{\mathrm{marg}}_{S_{-i}}\beta_i(t_i)$, and also (b) *every* strategy in $\mathrm{Proj}_{S_{-i}}(E)$ is infinitely more likely than (every strategy in) $S_{-i}\backslash\mathrm{Proj}_{S_{-i}}(E)$ under $\overline{\mathrm{marg}}_{S_{-i}}\beta_i(t_i)$. It should be noted that property (a) does not hold for (Weak) Assumption.[19]

The failure of one direction of conjunction reveals that, although "infinitely more likely" is monotone, Cautious Belief is not. That is, even if t_i cautiously believes E, t_i may not cautiously believe an event F with $E \subseteq F$. The reason why this can occur is that player i may not have towards F the same cautious attitude that she has towards E. That is, there may be some payoff-relevant components of $F\backslash E$ which are not deemed infinitely more likely than not-F.[20] This is illustrated by the following simple example.

Example 2 *Let $S_{-i} = \left\{s_{-i}^1, s_{-i}^2, s_{-i}^3\right\}$ and $T_{-i} = \left\{t_{-i}^*\right\}$. Consider the LPS $\overline{\mu}_i = (\mu_i^1, \mu_i^2) \in \mathcal{N}(S_{-i} \times T_{-i})$ with $\mu_i^1\left(\left\{\left(s_{-i}^1, t_{-i}^*\right)\right\}\right) = 1$ and $\mu_i^2\left(\left\{\left(s_{-i}^2, t_{-i}^*\right)\right\}\right) = \mu_i^2\left(\left\{\left(s_{-i}^3, t_{-i}^*\right)\right\}\right) = \frac{1}{2}$. Consider the events $E = \left\{s_{-i}^1\right\} \times T_{-i}$ and $F = \left\{s_{-i}^1, s_{-i}^2\right\} \times T_{-i}$. Clearly, $E \subseteq F$; however, E is cautiously believed under $\overline{\mu}_i$ at level 1, while F is not cautiously believed (indeed, $\mu_i^1(F) = 1$ and $\mu_i^2(F) = \frac{1}{2}$, and, with $l = 1$, condition (ii) of Definition 12 is not satisfied for $C = \left\{s_{-i}^2\right\} \times T_{-i}$).*

However, it is easy to observe that Cautious Belief is monotone with respect to events with the same behavioral implications.

Remark 4 *Let $E_{-i}, F_{-i} \subseteq S_{-i} \times T_{-i}$ be events such that $E_{-i} \subseteq F_{-i}$ and $\mathrm{Proj}_{S_{-i}}E_{-i} = \mathrm{Proj}_{S_{-i}}F_{-i}$. Then, if a type t_i cautiously believes E_{-i}, t_i also cautiously believes F_{-i}.*

This "quasi-monotonicity" property will play a crucial role in the proof of our main result.

For future reference, it is useful to mention the following notion of belief, called Certain Belief (Halpern 2010), which is stronger than Cautious Belief for cautious types. Fix a non-empty event $E \subseteq S_{-i} \times T_{-i}$ and a type $t_i \in T_i$ with $\beta_i(t_i) = (\mu_i^1, ..., \mu_i^n)$. We say that E is **certainly believed under** $\beta_i(t_i)$ if $\mu_i^l(E) = 1$ for all $l \leq n$. In other words, E is certainly believed under $\beta_i(t_i)$ if its complement is deemed subjectively impossible by the player (see Corollary A.1 for a preference-based foundation).

[19]If E is (weakly) assumed under $\beta_i(t_i)$, then condition (ii') holds (see BFK, Lemma D.1). However, $\mathrm{Proj}_{S_{-i}}(E)$ is not necessarily (weakly) assumed under $\overline{\mathrm{marg}}_{S_{-i}}\beta_i(t_i)$. This stems from the fact that the "infinitely more likely" notion of Blume et al. (1991a) may fail to satisfy a disjunction property, as illustrated by Example A.1 in Appendix A. If $\overline{\mathrm{marg}}_{S_{-i}}\beta_i(t_i)$ is a full-support LPS, then it is easy to show that $\mathrm{Proj}_{S_{-i}}(E)$ is "topologically weak-dominance assumed"—see the previous footnote.

[20]So, in Cautious Belief, non-monotonicity hinges *only* on the cautious attitude towards the event (namely, Condition (ii) of Definition 12). In a related vein, *Strong Belief* (Battigalli and Siniscalchi, 2002) is based on a monotone likelihood relation between uncertain events (conditional probability-one belief), but it does not satisfy monotonicity (we thank Pierpaolo Battigalli for this observation).

5 Epistemic analysis

5.1 Epistemic analysis of iterated admissibility

Fix a type structure $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i\in I}$ and, for each player $i \in I$, let $R_i^1 = R_i^c$. Let $\mathbf{B}_i^c : \Sigma_{S_{-i}\times T_{-i}} \to \Sigma_{S_i\times T_i}$ be the operator defined by

$$\mathbf{B}_i^c\left(E_{-i}\right) = \left\{(s_i, t_i) \in S_i \times T_i \,|t_i \text{ cautiously believes } E_{-i}\right\}, \, E_{-i} \in \Sigma_{S_{-i}\times T_{-i}}.$$

Corollary D.1 in Appendix D shows that the set $\mathbf{B}_i^c\left(E_{-i}\right)$ is Borel in $S_i \times T_i$; so the operator $\mathbf{B}_i^c$ is well-defined.

For each $m > 1$, define R_i^m inductively by

$$R_i^m = R_i^{m-1} \cap \mathbf{B}_i^c\left(R_{-i}^{m-1}\right).$$

Note that

$$R_i^m = R_i^1 \cap \left(\cap_{l<m}\mathbf{B}_i^c\left(R_{-i}^l\right)\right),$$

and each R_i^m is Borel in $S_i \times T_i$ (see Lemma D.5 in Appendix D).

We write $R_i^\infty = \cap_{m\in\mathbb{N}} R_i^m$ for each $i \in I$. If $(s_i, t_i)_{i\in I} \in \prod_{i\in I} R_i^{m+1}$, we say that there is **Cautious Rationality and mth-order Cautious Belief of Cautious Rationality** ($\mathbf{R}^c m\mathbf{B}^c\mathbf{R}^c$) at this state. If $(s_i, t_i)_{i\in I} \in \prod_{i\in I} R_i^\infty$, we say that there is **Cautious Rationality and Common Cautious Belief of Cautious Rationality** ($\mathbf{R}^c\mathbf{CB}^c\mathbf{R}^c$) at this state.

We now state the main result of this paper.

Theorem 1 *Fix a type structure $\mathcal{T}^* = \langle S_i, T_i^*, \beta_i^* \rangle_{i\in I}$ which is terminal with respect to the class of all finite type structures. Then:*

(i) *for each $m \geq 1$, $\prod_{i\in I} \mathrm{Proj}_{S_i} R_i^{*,m} = \prod_{i\in I} S_i^m$;*

(ii) *$\prod_{i\in I} R_i^{*,\infty} \neq \emptyset$ and $\prod_{i\in I} \mathrm{Proj}_{S_i} R_i^{*,\infty} = \prod_{i\in I} S_i^\infty$.*

Theorem 1 is the weak-dominance counterpart of Theorem 8.1 in Friedenberg and Keisler (2011). Both theorems characterize iterated dominance (respectively, weak and strong) in a terminal type structure for finite type structures. Hence, they provide a richness condition for the type structure that depends on its ability to capture sufficiently many hierarchies of beliefs (all those induced by finite type structures), as opposed to traditional self-referential conditions, such as belief-completeness.

It is important to point out that type structure $\mathcal{T}^*$ in Theorem 1 exists. In particular, there exists a *universal* type structure for LPS's, that is, a type structure which is terminal and for which the type morphism from every other type structure is unique.[21] Lee (2016b) shows the existence of a universal type structure for a wide class of preferences, which includes those represented by LPS's. Yang (2015) and Catonini and De Vito (2016)

[21] Within the framework of category theory, $(S_i)_{i\in I}$-based type structures for player set I, as objects, and type morphisms, as morphisms, form a category. The universal type structure is a terminal object in the category of type structures.

construct the canonical type structure for hierarchies of lexicographic beliefs; Catonini and De Vito also show that the this type structure is universal. Since the canonical type structure is continuous and belief-complete, it follows from Theorem 1 that $\mathrm{R^cCB^cR^c}$ is possible in a continuous, belief-complete type structure.

The proof of Theorem 1, like the proof of Theorem 8.1 in Friedenberg and Keisler (2011), is based on the following "embedding" argument. We construct a finite type structure $\mathcal{T}$ with one cautious type for each admissible strategy, where the associated LPS justifies the strategy and captures as many orders of Cautious Belief of Cautious Rationality as the number of steps of IA that the strategy survives *minus 1*. Then, by the terminality property of $\mathcal{T}^*$, we map $\mathcal{T}$ in $\mathcal{T}^*$ via type morphism. While doing so, we show that Cautious Rationality and all orders of belief of Cautious Rationality are preserved. For all this, we need the next three preparatory results, whose proofs can be found in Appendix C. First, we need to show the existence of $\mathcal{T}$.

Lemma 1 *There exists a finite type structure $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i \in I}$ such that, for each $i \in I$ and each $m \geq 1$, $\mathrm{Proj}_{S_i} R_i^m = S_i^m$.*

Second, we need to claim the invariance of Cautious Rationality under type morphisms.[22]

Lemma 2 *Let $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i \in I}$ and $\mathcal{T}^* = \langle S_i, T_i^*, \beta_i^* \rangle_{i \in I}$ be two lexicographic type structures. Suppose that there exists a type morphism $(\varphi_i)_{i \in I} : T \to T^*$ from $\mathcal{T}$ to $\mathcal{T}^*$. Then, a strategy-type pair (s_i, t_i) is cautiously rational if and only if $(s_i, \varphi_i(t_i))$ is cautiously rational.*

Third, we need an analogous invariance property for Cautious Belief.[23]

Lemma 3 *Let $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i \in I}$ and $\mathcal{T}^* = \langle S_i, T_i^*, \beta_i^* \rangle_{i \in I}$ be two lexicographic type structures. Suppose that there exists a bimeasurable type morphism $(\varphi_i)_{i \in I} : T \to T^*$ from $\mathcal{T}$ to $\mathcal{T}^*$. Let $E_{-i} \subseteq S_{-i} \times T_{-i}$ be a Borel set. Then, if a type $t_i \in T_i$ cautiously believes E_{-i}, $\varphi_i(t_i)$ cautiously believes $\left(\mathrm{Id}_{S_{-i}}, \varphi_{-i}\right)(E_{-i})$.*

For our purpose, it is crucial to observe that, by Remark 4, if $\varphi_i(t_i)$ cautiously believes $\left(\mathrm{Id}_{S_{-i}}, \varphi_{-i}\right)(E_{-i})$, then it cautiously believes also every Borel superset E_{-i}^* such that $\mathrm{Proj}_{S_{-i}} E_{-i}^* = \mathrm{Proj}_{S_{-i}} \left(\mathrm{Id}_{S_{-i}}, \varphi_{-i}\right)(E_{-i})$. Lemma 3 holds also for Weak Assumption (but not for Assumption) if $(\varphi_i)_{i \in I}$ is injective, but $\varphi_i(t_i)$ weakly assumes also a Borel superset

[22] This invariance property does not hold for the notion of Rationality *à la* BFK: If $\beta_i(t_i)$ has full support in a finite type structure $\mathcal{T}$, then $\beta_i^*(\varphi_i(t_i))$ has finite support as well, hence it is not a full-support LPS in $\mathcal{T}^*$ when T_{-i}^* is infinite.

[23] We thank an anonymous referee for suggesting to us the result as stated in Lemma 3.

E^*_{-i} only under some additional conditions.[24] Note, however, that we do not require that $(\varphi_i)_{i\in I}$ be injective to prove the main result.

Finally, for the proof of Theorem 1, we find it convenient to single out the following fact, whose proof is immediate.

Remark 5 *Let $\mathcal{T} = \langle S_i, T_i, \beta_i\rangle_{i\in I}$ and $\mathcal{T}^* = \langle S_i, T^*_i, \beta^*_i\rangle_{i\in I}$ be two lexicographic type structures. Suppose that there exists a type morphism $(\varphi_i)_{i\in I} : T \to T^*$ from $\mathcal{T}$ to $\mathcal{T}^*$. Then, for every $E_i \subseteq S_i \times T_i$,*

$$\mathrm{Proj}_{S_i}\left(\left(\mathrm{Id}_{S_i}, \varphi_i\right)(E_i)\right) = \mathrm{Proj}_{S_i}(E_i).$$

We are now ready to prove Theorem 1.

Proof of Theorem 1: By Lemma 1, there is a finite type structure $\mathcal{T} = \langle S_i, T_i, \beta_i\rangle_{i\in I}$ such that $\mathrm{Proj}_{S_i} R^m_i = S^m_i$ for each $m \geq 1$.

Part (i): Fix a type morphism $(\varphi_i)_{i\in I}$ from $\mathcal{T}$ to $\mathcal{T}^*$. Type structure $\mathcal{T}$ is finite, so $(\varphi_i)_{i\in I}$ is bimeasurable. We will show by induction on $m \geq 1$ that $\left(\mathrm{Id}_{S_i}, \varphi_i\right)(R^m_i) \subseteq R^{*,m}_i$ and $\mathrm{Proj}_{S_i} R^{*,m}_i = S^m_i$ for each $i \in I$.

($m = 1$) Fix $i \in I$. It is immediate from Lemma 2 that $\left(\mathrm{Id}_{S_i}, \varphi_i\right)(R^1_i) \subseteq R^{*,1}_i$. By Remark 5, $\mathrm{Proj}_{S_i}\left(\left(\mathrm{Id}_{S_i}, \varphi_i\right)(R^1_i)\right) = \mathrm{Proj}_{S_i} R^1_i$, and since $\mathrm{Proj}_{S_i} R^1_i = S^1_i$, we obtain $S^1_i \subseteq \mathrm{Proj}_{S_i} R^{*,1}_i$. Conversely, Proposition 1 entails $\mathrm{Proj}_{S_i} R^{*,1}_i \subseteq S^1_i$. Therefore, $\mathrm{Proj}_{S_i} R^{*,1}_i = S^1_i$.

($m > 1$) Fix $i \in I$ and $(s_i, t_i) \in R^m_i \subseteq R^{m-1}_i$. We want to show that $(s_i, \varphi_i(t_i)) \in R^{*,m}_i$. The induction hypothesis yields $(s_i, \varphi_i(t_i)) \in R^{*,m-1}_i$. Hence, it suffices to show that $\varphi_i(t_i)$ cautiously believes $R^{*,m-1}_{-i}$. Since t_i cautiously believes R^{m-1}_{-i}, it follows from Lemma 3 that $\varphi_i(t_i)$ cautiously believes $(\mathrm{Id}_{S_{-i}}, \varphi_{-i})(R^{m-1}_{-i})$. Next note that

$$\begin{aligned} \mathrm{Proj}_{S_{-i}} R^{*,m-1}_{-i} &= S^{m-1}_{-i} \\ &= \mathrm{Proj}_{S_{-i}} R^{m-1}_{-i} \\ &= \mathrm{Proj}_{S_{-i}}\left((\mathrm{Id}_{S_{-i}}, \varphi_{-i})(R^{m-1}_{-i})\right), \end{aligned}$$

where the first equality is the induction hypothesis, the second equality follows from the property of $\mathcal{T}$, and the third equality follows from Remark 5. We also know from the induction hypothesis that $\left(\mathrm{Id}_{S_{-i}}, \varphi_{-i}\right)(R^{m-1}_{-i}) \subseteq R^{*,m-1}_{-i}$, so Remark 4 allows us to conclude that $\varphi_i(t_i)$ cautiously believes $R^{*,m-1}_{-i}$.

So, we have shown that $\left(\mathrm{Id}_{S_i}, \varphi_i\right)(R^m_i) \subseteq R^{*,m}_i$. Then, by the property of $\mathcal{T}$ and Remark 5, we obtain

$$\begin{aligned} S^m_i &= \mathrm{Proj}_{S_i} R^m_i \\ &= \mathrm{Proj}_{S_i}\left(\left(\mathrm{Id}_{S_i}, \varphi_i\right)(R^m_i)\right) \\ &\subseteq \mathrm{Proj}_{S_i}\left(R^{*,m}_i\right). \end{aligned}$$

[24] For instance, every event $E^*_{-i} \supseteq \left(\mathrm{Id}_{S_{-i}}, \varphi_{-i}\right)(E_{-i})$ such that $\mathrm{Proj}_{S_i} E^*_{-i} = \mathrm{Proj}_{S_{-i}}\left(\mathrm{Id}_{S_{-i}}, \varphi_{-i}\right)(E_{-i})$ *and* $\left(\mathrm{Id}_{S_{-i}}, \varphi_{-i}\right)((S_{-i} \times T_{-i}) \backslash E_{-i}) \subseteq (S_{-i} \times T^*_{-i}) \backslash E^*_{-i}$ is weakly assumed by $\varphi_i(t_i)$ if t_i weakly assumes E_{-i}. However, it is possible to construct a more complicated finite type structure in such a way that R^m_i and $R^{*,m}_i$ satisfy this property. Hence, our argument for the proof of Theorem 1 can be modified with Weak Assumption in place of Cautious Belief.

To show the opposite inclusion, fix $(s_i, t_i) \in R_i^{*,m} \subseteq R_i^{*,m-1}$. Then $s_i \in S_i^{m-1}$. Let $\beta_i^*(t_i) = (\mu_i^1, ..., \mu_i^n)$. Since t_i cautiously believes $R_{-i}^{*,m-1}$ at some level l, it follows from Proposition 2.2 and the induction hypothesis that

$$\cup_{k \leq l} \text{Suppmarg}_{S_{-i}} \mu_i^k = S_{-i}^{m-1}.$$

So, by Proposition 1 in Blume et al. (1991b), we can form a nested convex combination of the measures $\text{marg}_{S_{-i}} \mu_i^k$, for $k = 1, ..., l$, to get a probability measure $\nu_i \in \mathcal{M}(S_{-i})$, with $\text{Supp}\nu_i = S_{-i}^{m-1}$, under which s_i is optimal. Thus, s_i is admissible with respect to $S_i \times S_{-i}^{m-1}$, and a fortiori with respect to $S_i^{m-1} \times S_{-i}^{m-1}$. Hence $s_i \in S_i^m$.

Part (ii): Since $(R_i^m)_{m\in\mathbb{N}}$ and $(S_i^m)_{m\in\mathbb{N}}$ are weakly decreasing sequences of finite sets, there exists $N \in \mathbb{N}$ such that $R_i^N = R_i^\infty$ and $S_i^N = S_i^\infty$. Then, $\text{Proj}_{S_i} R_i^N = S_i^N$ implies $\text{Proj}_{S_i} R_i^\infty = S_i^\infty$. Hence, for every $s_i \in S_i^\infty$, there exists $t_i \in T_i$ such that $(s_i, t_i) \in R_i^m$ for all $m \in \mathbb{N}$. We have shown in the proof of Part (i) that, for each $m \in \mathbb{N}$, $(\text{Id}_{S_i}, \varphi_i)(R_i^m) \subseteq R_i^{*,m}$. So $(\text{Id}_{S_i}, \varphi_i)((s_i, t_i)) \in R_i^{*,m}$ for all $m \in \mathbb{N}$, which implies that $(\text{Id}_{S_i}, \varphi_i)((s_i, t_i)) \in R_i^{*,\infty}$. Therefore, $S_i^\infty \subseteq \text{Proj}_{S_i} R_i^{*,\infty} \neq \emptyset$. By finiteness of the game, there exists $M \in \mathbb{N}$ such that $S_i^M = S_i^\infty$. It follows from Part (i) that $\text{Proj}_{S_i} R_i^{*,M} = S_i^M$. Hence $\text{Proj}_{S_i} R_i^{*,\infty} \subseteq S_i^\infty$. We can conclude that $S_i^\infty = \text{Proj}_{S_i} R_i^{*,\infty}$. ■

Comment on transparency of Cautiousness. Fix a type structure $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i\in I}$. Say that $\mathcal{T}$ is a **cautious type structure** if all the types of all players are cautious. In a cautious type structure, not only all the types are cautious, but there is common certain belief of this. In other words, there is **transparency** of Cautiousness. We let $C^\infty \subseteq \prod_{i\in I} S_i \times T_i$ denote the event corresponding to transparency of Cautiousness in $\mathcal{T}$, and we let C_i^∞ denote the corresponding projection on $S_i \times T_i$.[25] The finite type structure $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i\in I}$ we construct for Lemma 1 is cautious. Since Cautiousness is preserved by type morphisms (cf. Lemma 2) and the image of $\mathcal{T}$ in $\mathcal{T}^*$ via bimeasurable type morphism $(\varphi_i)_{i\in I}$ corresponds to a *self-evident event* in $\mathcal{T}^*$ (that is, an event in which there is common certain belief that the players' beliefs satisfy some restrictions—see Battigalli and Friedenberg 2012b, Appendix A), we have $\prod_{i\in I} \varphi_i(T_i) \subseteq C^{*,\infty}$. This implies that the proof of Theorem 1 can be easily adapted to provide two alternate justifications of IA that makes the transparency of Cautiousness fully explicit. First, suppose that the structure $\mathcal{T}^*$ in the statement of Theorem 1 is the canonical one, so that it is also universal. We can substitute the event "Cautious Rationality" with "Rationality and transparency of Cautiousness" in $\mathcal{T}^*$, that is, we can define $R_i^{*,1}$ as the set $R_i^{*,c} \cap C_i^{*,\infty}$ instead of just $R_i^{*,c}$. Then, Theorem 1 can be read as follows: For each $i \in I$, strategy s_i is m-admissible (resp. iteratively admissible) if and only if it can be played in some state $(s_i, t_i)_{i\in I}$ where

(a) players are rational and there is transparency of Cautiousness,

(b) there is mth-order (resp. Common) Cautious Belief of (a) at this state.

This is in the same spirit of the justification of Strong-Δ-Rationalizability (Battigalli 2003) provided by Battigalli and Prestipino (2013), where the Δ-restrictions on first-order beliefs are transparent to the players.

Alternatively, we can adopt a "smaller" terminal type structure, which is cautious and terminal with respect to the class of all cautious type structures. Fix any terminal type

[25] As shown in Appendix D, the set of cautious types of each $i \in I$ is a Borel subset of $S_i \times T_i$ (cf. Lemma D.4). The (Borel) set C^∞ can be defined by iterated application of the Certain Belief operator, using arguments and techniques analogous to those in Battigalli and Friedenberg (2012b, Lemma A1), and it can be espressed as a Cartesian product of Borel sets $C_i^\infty \subseteq S_i \times T_i$, $i \in I$.

structure $\mathcal{T}$. Using arguments analogous to those in Battigalli and Friedenberg (2012b, Appendix A), it is possible to show that the self-evident event C^{∞} identifies a "smaller" type structure which is terminal with respect to the class of cautious type structures, because, as observed, the image through type morphism of every cautious type structure is in C^{∞}. Of course, like in Theorem 1, we can just require terminality of $\mathcal{T}^*$ for the class of *finite* cautious type structures. This is in the same spirit of the justification of Strong-Δ-Rationalizability provided by Battigalli and Friedenberg (2012a), where transparency of the Δ-restrictions is a contextual assumption embodied in the type structure.

5.2 Epistemic analysis of Self-Admissible Sets

The following result states that, for every type structure, the behavioral implications of R^{c}CB^{c}R^c constitute a SAS. Conversely, every SAS corresponds to the behavioral implications of R^{c}CB^{c}R^c in some cautious type structure.

Theorem 2 *(i) Fix a type structure $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i \in I}$. Then, $\prod_{i \in I} \mathrm{Proj}_{S_i} R_i^{\infty}$ is an SAS.*
(ii) Fix an SAS $Q_i \times Q_{-i} \subseteq S_i \times S_{-i}$. There exists a finite, cautious type structure $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i \in I}$ such that, for each $i \in I$,

$$\mathrm{Proj}_{S_i} R_i^{\infty} = Q_i.$$

The proof of Theorem 2, which is provided in Appendix C, is very similar to BFK's proof of the justification of SAS with Rationality and Common Assumption of Rationality. Yet, there is an interesting difference in the proof of part (ii), which allows us to justify SAS within the class of cautious type structures. There is no need to introduce incautious types (i.e., types which are not cautious) in the finite type structure we construct. By contrast, the analysis in BFK requires the presence of types that are not associated with a full-support LPS. To see why this is the case, consider the following game (BFK, Figure 2.6, and Dekel and Siniscalchi 2015, Table 12.14) between two players, Ann (a) and Bob (b).

$Ann \backslash Bob$	L	C	R
U	$4,0$	$4,1$	$0,1$
M	$0,0$	$0,1$	$4,1$
D	$3,0$	$2,1$	$2,1$

Strategy L is strictly dominated, and the IA set is $\{U, M, D\} \times \{C, R\}$. Strategy D is best reply to a probability distribution over Bob's iteratively admissible strategies, namely $\frac{1}{2}C + \frac{1}{2}R$. But also U (as well as M) is a best reply to this (first-order) belief. Therefore, in every type structure, D is optimal under LPS $\beta_a(t_a) = (\mu_a^1, ..., \mu_a^n)$ only if for every $l = 1, ..., n$ such that $\mu_a^l(\{L\} \times T_b) > 0$, we also have $\mu_a^l(\{R\} \times T_b) > 0$. Since R is weakly dominant, if every type $t_b \in T_b$ is associated with a full-support LPS, all strategy-type pairs in $\{R\} \times T_b$ are rational *à la* BFK (see conditions R1 and R2 in the Introduction). Therefore, μ_a^l assigns positive probability both to the event "Bob is rational" and to its complement. But then, if $\beta_a(t_a)$ has full-support (so that there exists l with $\mu_a^l(\{L\} \times T_b) > 0$), type t_a cannot (weakly) assume rationality: see condition A1 in the Introduction. To circumvent this problem, BFK introduce a type $t_b \in T_b$ such that $\beta_b(t_b)$ is not of full-support.[26] Then, strategy D is optimal under some full-support LPS $\beta_a(t_a) = (\mu_a^1, ..., \mu_a^n)$ under which rationality is assumed at level m, and $\beta_a(t_a)$ assigns positive probability to the strategy-type pair (R, t_b) at some level $l > m$.

Differently from (Weak) Assumption, there are no restrictions on the component measures $\mu_a^{m+1}, ..., \mu_a^n$ of an LPS $\overline{\mu}_a = (\mu_a^1, ..., \mu_a^n)$ under which an event E is cautioulsy believed at level m. For this reason, we do not need incautious types and we can justify SAS and IA under transparency of Cautiousness.

5.3 Belief-completeness vs. terminality

The following result states that, for each non-degenerate game, there exists a continuous, belief-complete type structure where $\mathrm{R^cCB^cR^c}$ is not possible at any state.

Theorem 3 *Fix a game $G = \langle I, (S_i, u_i)_{i\in I}\rangle$ with $|S_i| \geq 2$ for each $i \in I$. There exists a continuous, belief-complete type structure $T = \langle S_i, T_i, \beta_i\rangle_{i\in I}$ such that*

$$\textstyle\prod_{i\in I} R_i^\infty = \emptyset.$$

Theorem 3 is proved in Appendix C. The proof is constructive. We let each type space be the Baire space, and we define a countable ordered partition $(\mathrm{T}_i^1, \mathrm{T}_i^2, ..)$ of it. Types in T_i^1 are not cautious, while all types in T_i^k $(k > 1)$ are cautious and are associated with all the possible LPS's that, at level 1, assign positive marginal probability to T_{-i}^{k-1}. This puts an upper bound on the strategic sophistication of the types in T_i^k, who cannot have $(k-1)$-th order Cautious Belief of Cautious Rationality.

Technically, the argument clearly exploits the lack of compactness of the type spaces, which allows strategic sophistication to grow indefinitely without ever reaching the infinite level. Conceptually, the reason why the set of states $\prod_{i\in I} R_i^\infty$ can be empty is the following: While a belief-complete type structure induces all beliefs about types, it need not induce all possible hierarchies of beliefs.

Theorems 1 and 3 imply that a belief-complete lexicographic type structure need not be terminal. In the context of ordinary probabilities (i.e., Subjective Expected Utility preferences) Friedenberg (2010, Theorem 3.1) shows that a belief-complete type structure is terminal provided each type space is compact and each belief map is continuous.[27] In the

[26] However, BFK also provide a conceptual justification for the introduction of types without full-support of the associated LPS. Full-support aims to capture the attitude of players who consider "everything possible," including the possibility that the opponent may not conform to this requirement. Seen from this angle, the fact that we do not need incautious types is conceptually consistent with our relaxation of the full-support requirement: Cautious players consider "possible" every payoff-relevant event, but not (necessarily) every event that pertains to the beliefs of the opponent. Hence, they need not consider the possibility that the opponent is not cautious. Note that Weak Assumption, although it relaxes the "caution" requirement towards the event, still requires types that are *not* associated with a full-support LPS for the characterization of IA (and SAS).

[27] The reverse implication is not true: A terminal type structure need not be belief-complete, unless the type structure is *belief-non-redundant* (Friedenberg 2010, Proposition 4.1), i.e., if distinct types induce distinct hierarchies of beliefs. This definition of belief-non-redundancy naturally extends to the case of

lexicographic case, there is no analogue of the aforementioned result. Catonini and De Vito (2016) show that a belief-complete type structure is *finitely terminal*, that is, following the terminology used by Friedenberg, it induces all finite-order beliefs. Such result is, in some sense, tight: Belief-completeness is insufficient to establish terminality, even though the continuity requirement on the belief maps is met. As already remarked (see Section 3.2), a belief-complete, lexicographic type structure cannot be compact and continuous; as such, Friedenberg's result cannot be extended to the lexicographic framework.

That said, it should be emphasized that BFK's impossibility result does not hinge on lack of terminality of the belief-complete type structure they construct. As shown by Keisler and Lee (2015), BFK's analysis depends on topological features of belief-complete type structures which are unrelated to belief hierarchies. By contrast, our message is in line with analogous works on other solution concepts, such as Iterated (Strict) Dominance: Friedenberg and Keisler (2011, Theorem 5.2) show that, for every non-degenerate finite game, there exists an associated belief-complete, standard type structure in which no strategy is consistent with Rationality and Common Belief of Rationality. They also show that this arises due to the lack of terminality of belief-complete type structures. Therefore, our negative result is an analogue of Friedenberg and Keisler's result in the lexicographic framework (and it follows some of their ideas for the construction of the type structure).

Appendix A: Preference-based representation of Cautious Belief

Fix a lexicographic type structure $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i \in I}$, where each strategy set S_i is finite. To ease notation, it will be convenient to set $\Omega = S_{-i} \times T_{-i}$ and to drop i's subscript from LPS's $\overline{\mu}_i$ on Ω.

An act on Ω is a Borel measurable function $f : \Omega \to [0,1]$. We let ACT(Ω) denote the set of all acts on Ω. A Decision Maker has preferences over elements of ACT(Ω). For $x \in [0,1]$, write $\overrightarrow{x}$ for the constant act associated with x, i.e., $\overrightarrow{x}(\omega) = x$ for all $\omega \in \Omega$. Each constant act is identified with the associated outcome in a natural way. In what follows, we assume that the outcome space $[0,1]$ is in utils, i.e., material consequences are replaced by their von Neumann-Morgenstern utility. Given a Borel set $E \subseteq \Omega$ and acts $f, g \in$ACT(Ω), define $(f_E, g_{\Omega \backslash E}) \in$ACT$(\Omega)$ as follows:

$$(f_E, g_{\Omega\backslash E})(\omega) = \begin{cases} f(\omega), & \omega \in E, \\ g(\omega), & \omega \in \Omega \diagdown E. \end{cases}$$

Let $\succsim$ be a preference relation on ACT(Ω) and write $\succ$ (resp. $\sim$) for strict preference (resp. indifference). The preference relation $\succsim$ satisfies the following axioms:

Axiom 1 Order: $\succsim$ is a complete, transitive, reflexive binary relation on ACT(Ω).

lexicographic type structures. Note, however, that this notion pertains to hierarchies of LPS's, not necessarily to hierarchies of lexicographic *preferences*. Multiple LPS's may represent the same lexicographic preference relation. See Lee (2013) for a detailed analysis of this issue.

Axiom 2 Independence: For all $f, g, h \in$ACT(Ω) and $\alpha \in (0,1]$,

$$\begin{aligned} f &\succ g \text{ implies } \alpha f + (1-\alpha)h \succ \alpha g + (1-\alpha)h, \text{ and} \\ f &\sim g \text{ implies } \alpha f + (1-\alpha)h \sim \alpha g + (1-\alpha)h. \end{aligned}$$

Moreover, let $\succsim_E$ denote the *conditional preference* given E, that is, $f \succsim_E g$ if and only if $(f_E, h_{\Omega\backslash E}) \succsim (g_E, h_{\Omega\backslash E})$ for some $h \in$ACT(Ω). Standard results (see Blume et al. 1991a for a proof) show that, under Axioms 1 and 2, $(f_E, h_{\Omega\backslash E}) \succsim (g_E, h_{\Omega\backslash E})$ holds for all $h \in$ACT(Ω) if it holds for some h.

Throughout, we maintain the assumption that $\overline{\mu}$ is a Lexicographic Expected Utility representation of $\succsim$, i.e., $\succsim = \succsim^{\overline{\mu}}$. (This makes sense, since each Lexicographic Expected Utility representation satisfies Axioms 1 and 2.) In what follows, we call $C \subseteq \Omega$ an **elementary cylinder** if $C = \{s_{-i}\} \times T_{-i}$ for some $s_{-i} \in S_{-i}$. Given s_{-i} and event E, we say that $E_{s_{-i}}$ is a **relevant part** of the event E if $E_{s_{-i}} = C \cap E \neq \emptyset$ for the elementary cylinder $C = \{s_{-i}\} \times T_{-i}$. Clearly, every non-empty event E can be written as a finite, disjoint union of all its relevant parts.

Definition A.1 *Say that $\succsim^{\overline{\mu}}$ exhibits **cautiousness** if, for every elementary cylinder $C = \{s_{-i}\} \times T_{-i}$, there are $f, g \in$ACT(Ω) such that $f \succ_C^{\overline{\mu}} g$.*

Recall that an event $E \subseteq \Omega$ is **Savage-null** under $\succsim$ if $f \sim_E g$ for all $f, g \in$ACT(Ω). Say that E is **non-null** under $\succsim$ if it is not Savage-null under $\succsim$. With this, we can introduce the notion of Certain Belief in terms of the preference relation $\succsim$.

Definition A.2 *Say that event $E \subseteq \Omega$ is **certainly believed** under $\succsim$ if $f \sim_{\Omega\backslash E} g$ for all $f, g \in$ACT(Ω).*

Savage-null events and Certain Belief can be characterized in terms of LPS's as follows.

Proposition A.1 *Fix $\overline{\mu} = (\mu^1, ..., \mu^n) \in \mathcal{N}(\Omega)$. An event $E \subseteq \Omega$ is Savage-null under $\succsim^{\overline{\mu}}$ if and only if $\mu^l(E) = 0$ for all $l \leq n$.*

Proof: If $\mu^l(E) = 0$ for all $l \leq n$, then obviously $f \sim_E^{\overline{\mu}} g$ for all $f, g \in$ACT(Ω). On the other hand, if $E \subseteq \Omega$ is Savage-null under $\succsim^{\overline{\mu}}$, then $\overrightarrow{1} \sim_E^{\overline{\mu}} \overrightarrow{0}$. That is,

$$\left(\int_E d\mu^l + \int_{\Omega\backslash E} f d\mu^l \right)_{l=1}^n = \left(0 + \int_{\Omega\backslash E} f d\mu^l \right)_{l=1}^n, \forall f \in \text{ACT}(\Omega),$$

which implies $\mu^l(E) = 0$ for all $l \leq n$. ■

Corollary A.1 *Fix $\overline{\mu} = (\mu^1, ..., \mu^n) \in \mathcal{N}(\Omega)$. A non-empty event $E \subseteq \Omega$ is certainly believed under $\succsim^{\overline{\mu}}$ if and only if it is certainly believed under $\overline{\mu}$.*

The set of *binary acts* (*bets*) on Ω is the set of all acts of the form $(\overrightarrow{x}_E, \overrightarrow{y}_{\Omega\backslash E})$, for $x, y \in [0,1]$ and event $E \subseteq \Omega$. As the rankings of binary acts reveal the Decision Maker's underlying beliefs or likelihoods, we introduce an "infinitely more likely" relation between events which is based on bets.

Definition A.3 *Fix events $E, F \subseteq \Omega$. Say that E is* ***more likely*** *than F under $\succsim^{\overline{\mu}}$ if for all $x, y \in [0,1]$ with $x > y$,*

$$(\overrightarrow{x}_E, \overrightarrow{y}_{\Omega\backslash E}) \succsim^{\overline{\mu}} (\overrightarrow{x}_F, \overrightarrow{y}_{\Omega\backslash F}).$$

Say that E is deemed ***infinitely more likely*** *than F under $\succsim^{\overline{\mu}}$, and write $E \gg^{\overline{\mu}} F$, if for all $x, y, z \in [0,1]$ with $x > y$,*

$$(\overrightarrow{x}_E, \overrightarrow{y}_{\Omega\backslash E}) \succ^{\overline{\mu}} (\overrightarrow{z}_F, \overrightarrow{y}_{\Omega\backslash F}).$$

In words, E is more likely than F if the Decision Maker prefers to bet on E rather than on F given the same prizes for the two bets; this choice-theoretic notion is due to Savage (1972, p. 31). On the other hand, E is infinitely more likely than F if betting on E is *strictly* preferable to betting on F, and strict preference persists no matter how bigger the prize z for winning the F bet is. This notion of "infinitely more likely" is due to Lo (1999, Definition 1).[28]

Note that, according to Definition A.3, if $E \gg^{\bar{\mu}} F$, then E is non-null under $\succsim^{\overline{\mu}}$, while F may, but *need not*, be Savage-null under $\succsim^{\overline{\mu}}$. When $\succsim^{\overline{\mu}}$ has a Subjective Expected Utility representation, $E \gg^{\bar{\mu}} F$ implies that F is Savage-null.

The likelihood relation $\gg^{\bar{\mu}}$ possesses some natural properties. First, it is irreflexive, asymmetric and transitive. Moreover, if $E \gg^{\bar{\mu}} F$, then

(P1) E is infinitely more likely than every Borel subset of F; and

(P2) every Borel superset of E is infinitely more likely than F.

We already referred to (P2) as monotonicity property in Section 4.2 (see Remark 2).

The next step is to characterize the likelihood order $\gg^{\bar{\mu}}$ between pairwise *disjoint* events in terms of LPS's representing $\succsim^{\overline{\mu}}$. This will lead us to the LPS-based Definition 11 of Section 4.2. Recall that, given an LPS $\overline{\mu} = (\mu^1, ..., \mu^n) \in \mathcal{N}(\Omega)$ and non-empty event $E \subseteq \Omega$,

$$\mathcal{I}_{\overline{\mu}}(E) = \inf \left\{ l \in \{1, ..., n\} \left| \mu^l(E) > 0 \right. \right\}.$$

Proposition A.2 *Fix $\overline{\mu} = (\mu^1, ..., \mu^n) \in \mathcal{N}(\Omega)$ and disjoint events $E, F \subseteq \Omega$ with $E \neq \emptyset$.*

1. *E is more likely than F under $\succsim^{\overline{\mu}}$ if and only if*

$$\left(\mu^l(E)\right)_{l=1}^n \geq_L \left(\mu^l(F)\right)_{l=1}^n.$$

2. *$E \gg^{\overline{\mu}} F$ if and only if $\mathcal{I}_{\overline{\mu}}(E) < \mathcal{I}_{\overline{\mu}}(F)$.*

Proof: The proof of part 1 is left to the reader. Part 2: The statement is clearly true if F is Savage-null under $\succsim^{\overline{\mu}}$, so that, by Proposition A.1, $\mathcal{I}_{\overline{\mu}}(F) = \inf \emptyset = +\infty$. So, in what follows, let F be non-null under $\succsim^{\overline{\mu}}$. Set $p = \mathcal{I}_{\overline{\mu}}(E)$ and $q = \mathcal{I}_{\overline{\mu}}(F)$ for convenience.

[28] Lo introduces such definition for a wide class of preferences, including the Lexicographic Expected Utility model.

(Necessity) Arguing contrapositively, suppose that $p \geq q$. We consider two cases:

(a) $p > q$. Let $x = z = 1$ and $y = 0$. We clearly have $(\overrightarrow{1}_F, \overrightarrow{0}_{\Omega\backslash F}) \succ^{\overline{\mu}} (\overrightarrow{1}_E, \overrightarrow{0}_{\Omega\backslash E})$, so $E \gg^{\overline{\mu}} F$ fails.

(b) $p = q$. Observe that, since E and F are disjoint, $\mu^q(E), \mu^q(F) \in (0,1)$. Let $x = \mu^q(F)$, $z = 1$ and $y = 0$. For all $l < q$, it holds that

$$\int (\overrightarrow{x}_E, \overrightarrow{0}_{\Omega\backslash E}) d\mu^l = \int (\overrightarrow{1}_F, \overrightarrow{0}_{\Omega\backslash F}) d\mu^l = 0,$$

while

$$\int (\overrightarrow{1}_F, \overrightarrow{0}_{\Omega\backslash F}) d\mu^q = \mu^q(F) > \mu^q(F) \cdot \mu^q(E) = \int (\overrightarrow{x}_E, \overrightarrow{0}_{\Omega\backslash E}) d\mu^q,$$

where the strict inequality follows from the observation above. Again, this shows that $E \gg^{\overline{\mu}} F$ fails.

(Sufficiency) Let $p < q$, and pick any $x, y, z \in [0,1]$ with $x > y$. For all $l < p$, it holds that

$$\int (\overrightarrow{x}_E, \overrightarrow{y}_{\Omega\backslash E}) d\mu^l = \int (\overrightarrow{z}_F, \overrightarrow{y}_{\Omega\backslash F}) d\mu^l = y.$$

Next note that, since $(x - y)\mu^p(E) > 0$,

$$x\mu^p(E) + y\mu^p(\Omega\backslash E) > y\mu^p(\Omega\backslash F) = y,$$

that is,

$$\int (\overrightarrow{x}_E, \overrightarrow{y}_{\Omega\backslash E}) d\mu^p > \int (\overrightarrow{z}_F, \overrightarrow{y}_{\Omega\backslash F}) d\mu^p.$$

This shows that $(\overrightarrow{x}_E, \overrightarrow{y}_{\Omega\backslash E}) \succ^{\overline{\mu}} (\overrightarrow{z}_F, \overrightarrow{y}_{\Omega\backslash F})$, as required. ■

Next the notion of Cautious Belief in terms of the likelihood order $\gg^{\overline{\mu}}$.

Definition A.4 *Fix $\overline{\mu} = (\mu^1, ..., \mu^n) \in \mathcal{N}(\Omega)$. A non-empty event $E \subseteq \Omega$ is* **cautiously believed** *under $\succsim^{\overline{\mu}}$ if it satisfies the following condition:*

(*) for every relevant part $E_{s_{-i}}$ of E, $E_{s_{-i}} \gg^{\overline{\mu}} \Omega\backslash E$.

That is, the event E is cautiously believed under $\succsim^{\overline{\mu}}$ if every relevant part of E is deemed infinitely more likely than $\Omega\backslash E$. Since E can be written as a finite, disjoint union of all its relevant parts, it follows from monotonicity of $\gg^{\overline{\mu}}$ (Property P2) that E is deemed infinitely more likely than $\Omega\backslash E$, i.e., $E \gg^{\overline{\mu}} \Omega\backslash E$. However, the opposite is not true. Indeed, Cautious Belief of E is stronger than requiring that $E \gg^{\overline{\mu}} \Omega\backslash E$,[29] as it captures cautious behavior. We can indeed formulate the following preference-based counterpart of Remark 3 in Section 4.2.

Remark A.1 *Fix $\overline{\mu} = (\mu^1, ..., \mu^n) \in \mathcal{N}(\Omega)$. The preference relation $\succsim^{\overline{\mu}}$ exhibits cautiousness if and only if Ω is cautiously believed under $\succsim^{\overline{\mu}}$.*

We now state and prove the characterization result for Cautious Belief. For the reader's convenience, we restate the LPS-based definition of Cautious Belief given in the main text, but in terms of relevant parts.

[29] If $E \gg^{\overline{\mu}} \Omega\backslash E$, we say that E is *weakly believed* under $\succsim^{\overline{\mu}}$. As is shown in Catonini and De Vito (2018), weak belief in E implies that $\mu^1(E) = 1$ for every $(\mu^1, ..., \mu^n) \in \mathcal{N}(\Omega)$ which represents $\succsim^{\overline{\mu}}$.

Definition A.5 *Fix $\overline{\mu} = (\mu^1, ..., \mu^n) \in \mathcal{N}(\Omega)$. A non-empty event $E \subseteq \Omega$ is* **cautiously believed under $\overline{\mu}$ at level** $m \leq n$ *if:*

(i) $\mu^l(E) = 1$ *for all* $l \leq m$*;*

(ii) *for every relevant part $E_{s_{-i}}$ of E, $\mu^l\left(E_{s_{-i}}\right) > 0$ for some $l \leq m$.*

Theorem A.2 *Fix $\overline{\mu} = (\mu^1, ..., \mu^n) \in \mathcal{N}(\Omega)$ and a non-empty event $E \subseteq \Omega$. Then E is cautiously believed under $\succsim^{\overline{\mu}}$ if and only if E is cautiously believed under $\overline{\mu}$.*

Proof: The proof is immediate if $\Omega \backslash E$ is Savage-null under $\succsim^{\overline{\mu}}$, so, in what follows, let $\Omega \backslash E$ be non-null under $\succsim^{\overline{\mu}}$.

(Necessity) Since every relevant part $E_{s_{-i}}$ of E satisfies $E_{s_{-i}} \gg^{\bar{\mu}} \Omega \backslash E$, Proposition A.2 yields $\mathcal{I}_{\overline{\mu}}\left(E_{s_{-i}}\right) < \mathcal{I}_{\overline{\mu}}(\Omega \backslash E)$. Hence, $\mathcal{I}_{\overline{\mu}}(\Omega \backslash E) > 1$. Let $m = \mathcal{I}_{\overline{\mu}}(\Omega \backslash E) - 1$. Then, $\mathcal{I}_{\overline{\mu}}\left(E_{s_{-i}}\right) \leq m$. Therefore condition (ii) of Definition A.5 is satisfied. Moreover, for every $k \leq m$, $\mu^k(\Omega \backslash E) = 0$, hence $\mu^k(E) = 1$. Thus, condition (i) of Definition A.5 is satisfied.

(Sufficiency) If E is cautiously believed under $\overline{\mu}$ at level m, then condition (i) of Definition A.5 implies $\mathcal{I}_{\overline{\mu}}(\Omega \backslash E) > m$. By this, condition (ii) yields that each $E_{s_{-i}}$ satisfies $\mathcal{I}_{\overline{\mu}}\left(E_{s_{-i}}\right) < \mathcal{I}_{\overline{\mu}}(\Omega \backslash E)$, hence, by Proposition A.2, $E_{s_{-i}} \gg^{\bar{\mu}} \Omega \backslash E$. ■

We now provide a brief comparison between the notion of "infinitely more likely" in Definition A.3 and the one of Blume et al. (1991a). Specifically, Blume et al. (1991a) examine a partial order $\gg_S^{\overline{\mu}}$ on events of Ω which is stronger than $\gg^{\bar{\mu}}$.

Definition A.6 *Fix $\overline{\mu} = (\mu^1, ..., \mu^n) \in \mathcal{N}(\Omega)$ and disjoint events $E, F \subseteq \Omega$ with $E \neq \emptyset$. Then, $E \gg_S^{\overline{\mu}} F$ if*

1 *E is non-null under $\succsim^{\overline{\mu}}$, and*

2 *for all $f, g \in \mathrm{ACT}(\Omega)$, $f \succ_E^{\overline{\mu}} g$ implies $f \succ_{E \cup F}^{\overline{\mu}} g$.*

Condition 2 in Definition A.6 states that, when comparing two acts f and g that give the same consequences in states not belonging to $E \cup F$, if $f \succ_E^{\overline{\mu}} g$, then the consequences in F "do not matter" for the strict preference $f \succ^{\overline{\mu}} g$.[30] In particular, if $F = \Omega \backslash E$, then condition 2 corresponds to "Strict Determination," which is part of the preference-based definition of (Weak) Assumption.

It is easy to check that if $E \gg_S^{\overline{\mu}} F$ then $E \gg^{\overline{\mu}} F$. The reverse implication is true provided both E and F are singleton sets. The key difference is represented by the following property:

[30] The definition of the partial order $\gg_S^{\overline{\mu}}$ is taken from Asheim and Sovik 2006 (p. 65). Definition 5.1 in Blume et al. (1991a) states that $E \gg_S^{\overline{\mu}} F$ if condition 2 in Definition A.6 is replaced by the following condition:

$$f \succ_E g \text{ implies } \left(f_{\Omega \backslash F}, h_F\right) \succ_{E \cup F} \left(g_{\Omega \backslash F}, h'_F\right)$$

for all $h, h' \in \mathrm{ACT}(\Omega)$. (Condition 1 is automatically satisfied in Definition 5.1 of Blume et al. 1991a, since the authors consider a finite state space without Savage-null events.) It is easy to check the equivalence between the two definitions.

Proposition A.3 *Fix $\overline{\mu} = (\mu^1, ..., \mu^n) \in \mathcal{N}(\Omega)$ and non-empty, pairwise disjoint events $E, F \subseteq \Omega$, with E non-null under $\succsim^{\overline{\mu}}$. The following property holds:*

(**) *Let $E_1 \subseteq \Omega$ be a non-empty event such that $E_1 \subseteq E$ and E_1 is non-null under $\succsim^{\overline{\mu}}$. If $E \gg_S^{\overline{\mu}} F$, then $E_1 \gg_S^{\overline{\mu}} F$.*

In words, Proposition A.3 states that $\gg_S^{\overline{\mu}}$ requires that each *non-null* Borel subset $E_1 \subseteq E$ be infinitely more likely than F. The cost is the non-monotonicity of the order $\gg_S^{\overline{\mu}}$. That is, if $E, F, G \subseteq \Omega$ are non-empty, pairwise disjoint events with $E \gg_S^{\overline{\mu}} F$, it may *not* be the case that $E \cup G \gg_S^{\overline{\mu}} F$.[31] This can be best seen by considering the example of Section 1.3, which shows that $\gg_S^{\overline{\mu}}$ fails monotonicity as well as disjunction (cf. Blume et al. 1991a, p. 70).

Example A.1: *Let $\overline{\mu}_a = (\mu_a^1, \mu_a^2)$ be the LPS as summarized in the following table.*

μ_a^1	t_b'	t_b''
u	$1/2$	$1/2$
d	0	0

μ_a^2	t_b'	t_b''
u	0	$1/2$
d	0	$1/2$

Consider the event $E = \{(u, t_b')\} \cup \{(u, t_b'')\}$, and take acts

$$\begin{aligned} f &= (\overrightarrow{2/3}_E, \overrightarrow{0}_{\Omega \backslash E}), \\ g &= \left(\overrightarrow{1}_{\{(u,t_b'),(d,t_b'')\}}, \overrightarrow{1/3}_{\Omega \backslash \{(u,t_b'),(d,t_b'')\}}\right). \end{aligned}$$

Note that $E \gg_S^{\overline{\mu}} \Omega \backslash E$ fails, because $f \succ_E^{\overline{\mu}} g$ while $g \succ^{\overline{\mu}} f$. However, it is immediate to check that $\{(u, t_b')\} \gg_S^{\overline{\mu}} \Omega \backslash E$ and $\{(u, t_b'')\} \gg_S^{\overline{\mu}} \Omega \backslash E$.

Additional perspective on the comparison between $\gg^{\overline{\mu}}$ and $\gg_S^{\overline{\mu}}$ can be provided by considering the alternative representation of LPS's in terms of infinitesimal non-standard real numbers. As is well known (see Blume et al. 1991a, Section 6), a preference relation admitting a Lexicographic Expected Utility representation can be equivalently described by an $\mathbb{F}$-valued probability measure on Ω, where $\mathbb{F}$ is a non-Archimedean ordered field which is a strict extension of the set of real numbers $\mathbb{R}$. For instance, the LPS $\overline{\mu}_a = (\mu_a^1, \mu_a^2)$ of Example A.1 can be represented by the non-standard real valued probability measure ν:

ν	t_b'	t_b''
u	$\frac{1}{2} - \varepsilon$	$\frac{1}{2}$
d	0	ε

Here $\varepsilon > 0$ is the infinitesimal non-standard real such that for each real number $x > 0$ and each $n \in \mathbb{N}$, it is the case that $x > n\varepsilon$.

Given non-standard reals x and y, say that x is **infinitely greater** than y if $x > ny$ for each $n \in \mathbb{N}$. So, as far as the probability measure ν is concerned, it can be easily seen that $\nu(\{(u, t_b')\} \cup \{(u, t_b'')\})$ is infinitely greater than $\nu(\{(d, t_b'')\})$. Note that $\{(u, t_b')\} \cup \{(u, t_b'')\} \gg^{\overline{\mu}} \{(d, t_b'')\}$. This is not a coincidence: the notion of infinitely more

[31] Yet, it can be easily shown that $E \cup G \gg_S^{\overline{\mu}} F$ provided G is Savage-null under $\succsim^{\overline{\mu}}$. In other words, the union of E with a non-null event can reduce the likelihood ranking of an event, while the union with a Savage-null event, paradoxically, cannot.

likely in Definition A.3 corresponds exactly to the "infinitely greater" relation between the non-standard probability values that provide an equivalent representation of preferences.[32] In light of Proposition A.3, the "infinitely more likely" notion of Blume et al. (1991a) is instead stronger than the "infinitely greater" relation: given events $E, F \subseteq \Omega$, if $E \gg_S^{\overline{\mu}} F$, then $\nu(E)$ is infinitely greater than $\nu(F)$ (and so every non-null Borel subset of E); but the reverse implication does not hold. Indeed, it is not true that $\{(u, t_b')\} \cup \{(u, t_b'')\} \gg_S^{\overline{\mu}} \{(d, t_b'')\}$ (see Example A.1).

Comment on Assumption. We conclude this section with the following remark. In the case of a finite space of uncertainty without Savage-null events, when preferences can be represented by a mutually singular LPS $\overline{\mu}$, the following statements are equivalent:
(a) $E \gg_S^{\overline{\mu}} \Omega \backslash E$;
(b) for every $\omega \in E$, $\{\omega\} \gg_S^{\overline{\mu}} \Omega \backslash E$;
(c) E is assumed under $\succeq^{\overline{\mu}}$.

But, as far as an infinite space Ω is concerned, Assumption needs to be stronger than (a) in order to provide an epistemic justification of IA (cf. BFK's Supplemental Appendix). Therefore, BFK let Assumption coincide with the following "infinite-analogue" of (b):
(b') for every relatively open subset O of E, $O \gg_S^{\overline{\mu}} \Omega \backslash E$.

Condition (b') is stronger than (a), because (a) only implies that the non-null Borel subsets of E be infinitely more likely than $\Omega \backslash E$. Therefore, under (a), it is important to specify a class of Borel subsets of E which must be non-null in order to capture IA. To obtain (b'), BFK impose the requirement that every relatively open subset of E be non-null (see their "Nontriviality" axiom).

Appendix B: Proofs for Section 4

We begin with the proof of Proposition 1.

Proof of Proposition 1: By definition, if $(s_i, t_i) \in R_i^c$, then s_i is a lexicographic best reply to $\overline{\mathrm{marg}}_{S_{-i}} \beta_i(t_i) \in \mathcal{N}^+(S_{-i})$. Proposition 1 in Blume et al. (1991b) states that for every $\overline{\mu}_i \in \mathcal{N}^+(S_{-i})$ and every lexicographic best reply s_i' to $\overline{\mu}_i$, there exists a probability measure $\nu_i \in \mathcal{M}(S_{-i})$ such that $\mathrm{Supp}\nu_i = S_{-i}$ and $\pi_i(s_i', \nu_i) \geq \pi_i(s_i'', \nu_i)$ for every $s_i'' \in S_i$. Thus, s_i is admissible. ■

We next prove Proposition 2. To this end, we find it convenient to state and prove an auxiliary result, which is the analogue of Lemma B.1 in BFK.

Lemma B.1 *Fix a type $t_i \in T_i$ with $\beta_i(t_i) = (\mu_i^1, ..., \mu_i^n)$ and a non-empty event $E \subseteq S_{-i} \times T_{-i}$. Then, E is cautiously believed under $\beta_i(t_i)$ if and only if there exists $m \leq n$ such that $\beta_i(t_i)$ satisfies condition (i) of Definition 12 plus the following condition:*
(ii") $E \subseteq \left(\cup_{l \leq m} \mathrm{Suppmarg}_{S_{-i}} \mu_i^l\right) \times T_{-i}$.

[32] We thank an anonymous referee for this observation.

Proof: Suppose that E is cautiously believed under $\beta_i(t_i) = (\mu_i^1, ..., \mu_i^n)$ at level m. We show that $\beta_i(t_i)$ satisfies condition (ii"). For every $s_{-i} \in \text{Proj}_{S_{-i}}(E)$, we have

$$(\{s_{-i}\} \times T_{-i}) \cap E \neq \emptyset.$$

Hence, by condition (ii) of Definition 12, there exists $k \leq m$ such that $\mu_i^k(\{s_{-i}\} \times T_{-i}) > 0$. Thus, $s_{-i} \in \text{Suppmarg}_{S_{-i}}\mu_i^k$. So we obtain

$$\begin{aligned} E &\subseteq \text{Proj}_{S_{-i}}(E) \times T_{-i} \\ &\subseteq \left(\cup_{l \leq m} \text{Suppmarg}_{S_{-i}}\mu_i^l\right) \times T_{-i}. \end{aligned}$$

Conversely, suppose that conditions (i) and (ii") hold. We show that condition (ii) of Definition 12 holds. Fix $s_{-i} \in S_{-i}$ such that $E_{s_{-i}} = (\{s_{-i}\} \times T_{-i}) \cap E \neq \emptyset$. By condition (ii"), $E_{s_{-i}} \subseteq \left(\cup_{l \leq m} \text{Suppmarg}_{S_{-i}}\mu_i^l\right) \times T_{-i}$. Hence there exists $k \leq m$ such that $s_{-i} \in \text{Suppmarg}_{S_{-i}}\mu_i^k$. So $\mu_i^k(\{s_{-i}\} \times T_{-i}) > 0$. Moreover, by condition (i), $\mu_i^k(E) = 1$. Theredore $\mu_i^k\left(E_{s_{-i}}\right) > 0$, as desired. ■

Proof of Proposition 2: Part 1: Let $\overline{\mu}_i = (\mu_i^1, ..., \mu_i^n)$ and suppose that, for each k, E_k is cautiously believed under $\overline{\mu}_i$ at some level m_k. Let $m_K = \min\{m_k \,|k = 1, 2, ...\}$. We show that $E = \cap_k E_k$ is cautiously believed at level m_K. For each k, it holds that $\mu_i^l(E_k) = 1$ for all $l \leq m_K$. By the σ-additivity property of probability measures, it follows that $\mu_i^l(E) = 1$ for all $l \leq m_K$. Fix an elementary cylinder $C = \{s_{-i}\} \times T_{-i}$ with $E \cap C \neq \emptyset$. Obviously, $E_{m_K} \cap C \neq \emptyset$. Since E_{m_K} is cautiously believed at level m_K, by condition (ii) of Definition 12 we have $\mu_i^l(E_{m_K} \cap C) > 0$ for some $l \leq m_K$. Since $\mu_i^l(E) = 1$, we obtain

$$0 < \mu_i^l(E_{m_K} \cap C) = \mu_i^l(E_{m_K} \cap C \cap E) \leq \mu_i^l(E \cap C).$$

Now, let $m_K = \max\{m_k \,|k = 1, 2, ...\}$. We show that $E = \cup_k E_k$ is cautiously believed at level m_K. For each $l \leq m_K$, $1 = \mu_i^l(E_{m_K}) \leq \mu_i^l(E)$. For each elementary cylinder $C = \{s_{-i}\} \times T_{-i}$ with $E \cap C \neq \emptyset$, there is k such that $E_k \cap C \neq \emptyset$. By condition (ii) of Definition 12, it follows that $0 < \mu_i^l(E_k \cap C) \leq \mu_i^l(E \cap C)$ for some $l \leq m_k \leq m_K$.

Part 2: Suppose that condition (i) of Definition 12 and condition (ii') are satisfied. Then condition (ii') implies

$$\begin{aligned} E &\subseteq \text{Proj}_{S_{-i}}^{-1}\left(\text{Proj}_{S_{-i}}(E)\right) \\ &= \text{Proj}_{S_{-i}}^{-1}\left(\left(\cup_{l \leq m} \text{Suppmarg}_{S_{-i}}\mu_i^l\right)\right) \\ &= \left(\cup_{l \leq m} \text{Suppmarg}_{S_{-i}}\mu_i^l\right) \times T_{-i}, \end{aligned}$$

i.e., condition (ii") in Lemma B.1 holds. Hence E is cautiously believed under $\beta_i(t_i)$.

For the converse, suppose that E is cautiously believed under $\beta_i(t_i) = (\mu_i^1, ..., \mu_i^n)$ at level m. By Lemma B.1, it follows that

$$\begin{aligned} \text{Proj}_{S_{-i}}(E) &\subseteq \text{Proj}_{S_{-i}}\left(\left(\cup_{l \leq m} \text{Suppmarg}_{S_{-i}}\mu_i^l\right) \times T_{-i}\right) \\ &= \cup_{l \leq m} \text{Suppmarg}_{S_{-i}}\mu_i^l. \end{aligned}$$

To show that this set inclusion holds with equality, let $s_{-i} \notin \text{Proj}_{S_{-i}}(E)$. Then $(\{s_{-i}\} \times T_{-i}) \cap E = \emptyset$. By condition (i) of Definition 12, $\mu_i^l(E) = 1$ for each $l \leq m$, so

$$\mu_i^l(\{s_{-i}\} \times T_{-i}) = \text{marg}_{S_{-i}}\mu_i^l(\{s_{-i}\}) = 0.$$

This implies $s_{-i} \notin \text{Suppmarg}_{S_{-i}}\mu_i^l$. ■

Appendix C: Proofs for Section 5

In this section we provide the proofs of Lemmas 1-3, as well as the proofs of Theorem 2 and Theorem 3.

Proof of Lemma 1: Let $M \geq 1$ be the smallest natural number such that $\prod_{i \in I} S_i^\infty = \prod_{i \in I} S_i^M$.[33] By Lemma E.1 in BFK, for every $n \in \{1, ..., M+1\}$ and $s_i \in S_i^n$, there exists $\mu_{s_i}^n \in \mathcal{M}(S_{-i})$ such that $\mathrm{Supp}\mu_{s_i}^n = S_{-i}^{n-1}$ and

$$\pi_i(s_i, \mu_{s_i}^n) \geq \pi_i(s_i', \mu_{s_i}^n), \forall s_i' \in S_i.$$

We use this result to construct a finite type structure $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i \in I}$ as follows.

For each $i \in I$, let $T_i = S_i^1$, and define each belief map $\beta_i : T_i \to \mathcal{N}(S_{-i} \times T_{-i})$ as follows. Let $s_i \in T_i$. Fix an arbitrary $\bar{s}_{-i} \in T_{-i}$ and define $\nu_{s_i}^1 \in \mathcal{M}(S_{-i} \times T_{-i})$ as

$$\nu_{s_i}^1 \left(\{(s_{-i}, \bar{s}_{-i})\} \right) = \mu_{s_i}^1 \left(s_{-i} \right), \forall s_{-i} \in S_{-i}.$$

Let $m = \max \left\{ k \leq M+1 | s_i \in S_i^k \right\}$. (Note that if $s_i \in S_i^M$, then $m = M+1$, because $S_i^M = S_i^{M+1}$.) So, if $m = 1$, let $\beta_i(s_i) = \left(\nu_{s_i}^1 \right)$. Otherwise, for each $k = 2, ..., m$, define $\nu_{s_i}^k \in \mathcal{M}(S_{-i} \times T_{-i})$ as

$$\nu_{s_i}^k \left(\{(s_{-i}, s_{-i})\} \right) = \mu_{s_i}^k \left(s_{-i} \right), \forall s_{-i} \in S_{-i}^{k-1},$$

and let

$$\beta_i(s_i) = \left(\nu_{s_i}^m, ..., \nu_{s_i}^1 \right).$$

Finiteness of each type set guarantees that each belief map is Borel measurable (in fact, continuous). This completes the definition of the type structure $\mathcal{T}$.

We now show that $\mathcal{T}$ satisfies the required properties. To this end, we find it convenient to define, for each $i \in I$ and $k = 1, ..., M$, the following sets:

$$\Delta_{S_i^k \times T_i} = \left\{ (s_i, s_i') \in S_i^k \times T_i | s_i = s_i' \right\}.$$

That is, each set $\Delta_{S_i^k \times T_i}$ is homeomorphic to the diagonal of $S_i^k \times S_i^k$.[34] Next note that, for every $s_i \in S_i^2$, all the component measures of $\beta_i(s_i) = \left(\nu_{s_i}^m, ..., \nu_{s_i}^1 \right)$ except for $\nu_{s_i}^1$ are concentrated on those "diagonal" sets, namely

$$\mathrm{Supp}\nu_{s_i}^k = \Delta_{S_{-i}^{k-1} \times T_{-i}}, \ k = 2, ..., m,$$

which implies $\mathrm{Supp}\nu_{s_i}^k \subseteq \mathrm{Supp}\nu_{s_i}^{k-1}$ for $k \geq 3$.

The rest of the proof is by induction.

Induction Hypothesis (n): For each $i \in I$, $\mathrm{Proj}_{S_i} R_i^n = S_i^n$; moreover, $\Delta_{S_i^n \times T_i} \subseteq R_i^n$ if $n \leq M$, and $\Delta_{S_i^M \times T_i} \subseteq R_i^n$ if $n > M$.

[33] Note that, if $S^0 = S^1$, then M is 1 and not 0. This will simplify exposition.

[34] The diagonal of $S_i^k \times S_i^k$ is the set

$$\left\{ (s_i, s_i') \in S_i^k \times S_i^k | s_i = s_i' \right\}.$$

Basis Step ($n=1$). Fix $i \in I$ and $s_i \in S_i^1$. Type s_i is cautious, because

$$\text{Suppmarg}_{S_{-i}}\nu_{s_i}^1 = \text{Supp}\mu_{s_i}^1 = S_{-i},$$

and the strategy-type pair (s_i, s_i) is rational, in that

$$\overline{\text{marg}}_{S_{-i}}\beta_i(s_i) = \left(\mu_{s_i}^m, ..., \mu_{s_i}^1\right).$$

This shows that $(s_i, s_i) \in R_i^1$. Therefore $\Delta_{S_i^1 \times T_i} \subseteq R_i^1$, which implies that $S_i^1 \subseteq \text{Proj}_{S_i} R_i^1$. Conversely, Proposition 1 yields $\text{Proj}_{S_i} R_i^1 \subseteq S_i^1$.

Inductive Step ($n+1$). For each $i \in I$, we have to show the following: (1) $\text{Proj}_{S_i} R_i^{n+1} = S_i^{n+1}$; (2) $\Delta_{S_i^{n+1} \times T_i} \subseteq R_i^{n+1}$ if $n+1 \leq M$, and $\Delta_{S_i^M \times T_i} \subseteq R_i^{n+1}$ if $n+1 > M$.

Fix $i \in I$ and $s_i \in S_i^{n+1}$. Let $k = \min\{n+1, M+1\}$. Since $(s_i, s_i) \in \Delta_{S_i^{k-1} \times T_i}$, by the induction hypothesis it follows that $(s_i, s_i) \in R_i^n$. We show that $(s_i, s_i) \in \mathbf{B}_i^c\left(R_{-i}^n\right)$; this will yield $(s_i, s_i) \in R_i^{n+1}$. Write $\beta_i(s_i) = \left(\nu_{s_i}^m, ..., \nu_{s_i}^1\right)$, where $m \geq k$ because $s_i \in S_i^k$ and so, by construction, $\beta_i(s_i)$ must have length at least k. To show that R_{-i}^n is cautiously believed under $\beta_i(s_i)$, recall that $\text{Supp}\nu_{s_i}^l = \Delta_{S_{-i}^{l-1} \times T_{-i}} \subseteq \Delta_{S_{-i}^{k-1} \times T_{-i}}$ for each $l = k, ..., m$. Since $\Delta_{S_{-i}^{k-1} \times T_{-i}} \subseteq R_{-i}^n$ (induction hypothesis), it follows that condition (i) of Definition 12 is satisfied at level $l = m - k + 1$. Recall also that $\text{Supp}\nu_{s_i}^k = \Delta_{S_{-i}^{k-1} \times T_{-i}}$. By the induction hypothesis, $\text{Proj}_{S_{-i}} R_{-i}^n = S_{-i}^{k-1} = \text{Proj}_{S_{-i}} \Delta_{S_{-i}^{k-1} \times T_{-i}}$. Hence, $\beta_i(s_i)$ satisfies also condition (ii') of Proposition 2.2. Thus $(s_i, s_i) \in \mathbf{B}_i^c\left(R_{-i}^n\right)$, as required. So, we have shown that $S_i^{n+1} \subseteq \text{Proj}_{S_i} R_i^{n+1}$. For part (2), note the following fact: If $n+1 \leq M$, for every $(s_i, s_i) \in \Delta_{S_i^{n+1} \times T_i}$ we have $s_i \in S_i^{n+1}$; analogously, if $n+1 > M$, for every $(s_i, s_i) \in \Delta_{S_i^M \times T_i}$ we have $s_i \in S_i^{n+1}$. Then, by proving that $(s_i, s_i) \in R_i^{n+1}$ for each $s_i \in S_i^{n+1}$, we have proven (2).

Conversely, pick any $(s_i, s_i') \in R_i^{n+1} \subseteq R_i^n$. Then, by the induction hypothesis, $s_i \in S_i^n$. Let $\beta_i(s_i') = (\mu^1, ..., \mu^n)$. Since s_i' cautiously believes R_{-i}^n at some level l, it follows from Proposition 2.2 and the induction hypothesis that

$$\cup_{k \leq l} \text{Suppmarg}_{S_{-i}} \mu^k = S_{-i}^n.$$

So, by Proposition 1 in Blume et al. (1991b), there exists $\nu \in \mathcal{M}(S_{-i})$ with $\text{Supp}\nu = S_{-i}^n$ under which s_i is optimal. Therefore $s_i \in S_i^{n+1}$. This shows that $\text{Proj}_{S_i} R_i^{n+1} \subseteq S_i^{n+1}$, establishing (1). ■

Proof of Lemma 2: Fix a type $t_i \in T_i$, and let $\beta_i(t_i) = \left(\beta_i^1(t_i), ..., \beta_i^n(t_i)\right)$ be the associated LPS. Let O be a non-empty subset of S_{-i}. If $\overline{\text{marg}}_{S_{-i}}\beta_i(t_i) \in \mathcal{N}^+(S_{-i})$, then there is $l \leq n$ such that $\beta_i^l(t_i)(O \times T_{-i}) > 0$. It follows from the definition of type morphism that $\overline{\text{marg}}_{S_{-i}}\beta_i^*(\varphi_i(t_i)) \in \mathcal{N}^+(S_{-i})$, since

$$\begin{aligned} \beta_i^{*,l}(\varphi_i(t_i))\left(O \times T_{-i}^*\right) &= \beta_i^l(t_i)\left(\left(\text{Id}_{S_{-i}}, \varphi_{-i}\right)^{-1}\left(O \times T_{-i}^*\right)\right) \\ &= \beta_i^l(t_i)(O \times T_{-i}) > 0. \end{aligned}$$

An analogous argument shows that the reverse implication is also true. ■

Proof of Lemma 3: Fix a type t_i that cautiously believes E_{-i}. Let $t_i^* = \varphi_i(t_i)$. Note that bimeasurability of $(\varphi_i)_{i \in I}$ implies that $\left(\text{Id}_{S_{-i}}, \varphi_{-i}\right)(E_{-i})$ is an event in $S_{-i} \times T_{-i}$. We show that t_i^* cautiously believes $\left(\mathrm{Id}_{S_{-i}}, \varphi_{-i}\right)(E_{-i})$, that is, $\beta_i^*(t_i^*)$ satisfies conditions (i) and (ii) of Definition 12.

First, note that

$$E_{-i} \subseteq \left(\mathrm{Id}_{S_{-i}}, \varphi_{-i}\right)^{-1}\left(\left(\mathrm{Id}_{S_{-i}}, \varphi_{-i}\right)(E_{-i})\right).$$

Hence, by definition of type morphism, it follows that, for all $l \leq n$,

$$\begin{aligned} \beta_i^l(t_i)(E_{-i}) &\leq \beta_i^l(t_i)\left(\left(\mathrm{Id}_{S_{-i}}, \varphi_{-i}\right)^{-1}\left(\left(\mathrm{Id}_{S_{-i}}, \varphi_{-i}\right)(E_{-i})\right)\right) \\ &= \beta_i^{*,l}(t_i^*)\left(\left(\mathrm{Id}_{S_{-i}}, \varphi_{-i}\right)(E_{-i})\right). \end{aligned}$$

Since E_{-i} is cautiously believed under $\beta_i(t_i) = \left(\beta_i^1(t_i), ..., \beta_i^n(t_i)\right)$, it follows from condition (i) of Definition 12 that there exists $m \leq n$ such that $\beta_i^l(t_i)(E_{-i}) = 1$ for all $l \leq m$. Therefore, we have that $\beta_i^{*,l}(t_i^*)\left(\left(\mathrm{Id}_{S_{-i}}, \varphi_{-i}\right)(E_{-i})\right) = 1$ for all $l \leq m$. Hence $\beta_i^*(t_i^*)$ satisfies condition (i) of Definition 12.

Consider now an elementary cylinder $C = \{s_{-i}\} \times T_{-i}^*$ satisfying $\left(\mathrm{Id}_{S_{-i}}, \varphi_{-i}\right)(E_{-i}) \cap C \neq \emptyset$. First, note that

$$\begin{aligned} (\{s_{-i}\} \times T_{-i}) \cap E_{-i} &\subseteq (\{s_{-i}\} \times T_{-i}) \cap \left(\left(\mathrm{Id}_{S_{-i}}, \varphi_{-i}\right)^{-1}\left(\left(\mathrm{Id}_{S_{-i}}, \varphi_{-i}\right)(E_{-i})\right)\right) \\ &= \left(\left(\mathrm{Id}_{S_{-i}}, \varphi_{-i}\right)^{-1}(C)\right) \cap \left(\left(\mathrm{Id}_{S_{-i}}, \varphi_{-i}\right)^{-1}\left(\left(\mathrm{Id}_{S_{-i}}, \varphi_{-i}\right)(E_{-i})\right)\right) \\ &= \left(\mathrm{Id}_{S_{-i}}, \varphi_{-i}\right)^{-1}\left(C \cap \left(\mathrm{Id}_{S_{-i}}, \varphi_{-i}\right)(E_{-i})\right). \end{aligned}$$

Hence, by definition of type morphism, it follows that, for all $l \leq n$,

$$\begin{aligned} \beta_i^l(t_i)((\{s_{-i}\} \times T_{-i}) \cap E_{-i}) &\leq \beta_i^l(t_i)\left(\left(\mathrm{Id}_{S_{-i}}, \varphi_{-i}\right)^{-1}\left(C \cap \left(\mathrm{Id}_{S_{-i}}, \varphi_{-i}\right)(E_{-i})\right)\right) \\ &= \beta_i^{*,l}(t_i^*)\left(C \cap \left(\mathrm{Id}_{S_{-i}}, \varphi_{-i}\right)(E_{-i})\right). \end{aligned}$$

Since E_{-i} is cautiously believed under $\beta_i(t_i)$ at level $m \leq n$, and since $C \cap \left(\mathrm{Id}_{S_{-i}}, \varphi_{-i}\right)(E_{-i}) \neq \emptyset$ implies $(\{s_{-i}\} \times T_{-i}) \cap E_{-i} \neq \emptyset$, by condition (ii) of Definition 12 there exists $k \leq m$ such that $\beta_i^k(t_i)((\{s_{-i}\} \times T_{-i}) \cap E_{-i}) > 0$. Therefore, we have that

$$\beta_i^{*,k}(t_i^*)\left(C \cap \left(\mathrm{Id}_{S_{-i}}, \varphi_{-i}\right)(E_{-i})\right) > 0.$$

Thus, $\beta_i^*(t_i^*)$ satisfies condition (ii) of Definition 12. ■

Proof of Theorem 2: Part (i): Fix a type structure $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i \in I}$. If $\prod_{i \in I} \mathrm{Proj}_{S_i} R_i^{\infty} = \emptyset$, the result is immediate. So in what follows we will assume that this set is non-empty. For each $i \in I$ and $s_i \in \mathrm{Proj}_{S_i} R_i^{\infty}$, there exists $t_i \in T_i$ such that $(s_i, t_i) \in R_i^{\infty}$. Since $(s_i, t_i) \in R_i^1$, it follows that s_i is a lexicographic best reply to $\overline{\mathrm{marg}}_{S_{-i}} \beta_i(t_i) \in \mathcal{N}^+(S_{-i})$. Therefore, by Proposition 1, s_i is admissible, hence condition (a) of Definition 3 is satisfied. Next note that, for each $k \geq 1$, type t_i cautiously believes R_{-i}^k. So, it follows from Proposition 2.1 that R_{-i}^{∞} is cautiously believed under $\beta_i(t_i) = (\mu^1, ..., \mu^n)$ at some level m. Moreover, Proposition 2.2 entails $\cup_{l \leq m} \mathrm{Supp}\mathrm{marg}\mu^l = \mathrm{Proj}_{S_{-i}} R_{-i}^{\infty}$. Since s_i is a lexicographic best reply to $\overline{\mathrm{marg}}_{S_{-i}} \beta_i(t_i)$, Proposition 1 in Blume et al. (1991b) yields the existence of some $\nu \in \mathcal{M}(S_{-i})$ under which s_i is optimal and such that $\mathrm{Supp}\nu = \mathrm{Proj}_{S_{-i}} R_{-i}^{\infty}$. This shows that s_i is admissible with respect to $S_i \times \mathrm{Proj}_{S_{-i}} R_{-i}^{\infty}$, establishing condition (b) of Definition 3. Finally, by Corollary A1 in Brandenburger and Friedenberg (2010), every s_i' that supports s_i is a lexicographic best reply to $\overline{\mathrm{marg}}_{S_{-i}}\beta_i(t_i)$ as well. It follows that $(s_i', t_i) \in R_i^\infty$, and this in turn implies that $s_i' \in \mathrm{Proj}_{S_i} R_i^\infty$, establishing condition (c) of Definition 3.

Part (ii): Let $Q_i \times Q_{-i}$ be a non-empty SAS. Fix $i \in I$ and $s_i \in Q_i$. By conditions (a) and (b) of Definition 3, there exist $\nu_{s_i}^2, \nu_{s_i}^1 \in \mathcal{M}(S_{-i})$ such that $\mathrm{Supp}\nu_{s_i}^2 = S_{-i}$ and $\mathrm{Supp}\nu_{s_i}^1 = Q_{-i}$, and such that s_i is optimal under $\nu_{s_i}^2$ and $\nu_{s_i}^1$. Hence, s_i is a lexicographic best reply to $(\nu_{s_i}^1, \nu_{s_i}^2) \in \mathcal{N}^+(S_{-i})$. Moreover, as in BFK (p. 328), we can choose $\nu_{s_i}^2$ and $\nu_{s_i}^1$ in such a way that every strategy s_i' is optimal under $\nu_{s_i}^2$ and $\nu_{s_i}^1$ if and only if s_i' is supported by s_i. Now we construct a finite type structure $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i \in I}$ as follows.

For each $i \in I$, let $T_i = Q_i$. For every $s_i \in T_i$, define $\mu_{s_i}^1, \mu_{s_i}^2 \in \mathcal{M}(S_{-i} \times T_{-i})$ as

$$\begin{aligned} \mu_{s_i}^1(\{(s_{-i}, s_{-i})\}) &= \nu_{s_i}^1(s_{-i}), \forall s_{-i} \in Q_{-i}, \\ \mu_{s_i}^2(\{s_{-i}, \bar{s}_{-i}\}) &= \nu_{s_i}^2(s_{-i}), \forall s_{-i} \in S_{-i}, \end{aligned}$$

where $\bar{s}_{-i} \in T_{-i}$ is arbitrarily chosen.[35] Let $\beta_i(s_i) = (\mu_{s_i}^1, \mu_{s_i}^2)$. Finiteness of each type set guarantees that each belief map is measurable (in fact, continuous). This completes the definition of the type structure $\mathcal{T}$.

We now show that $\mathcal{T}$ satisfies the required properties. Note that each type $s_i \in T_i$ is cautious because $\mathrm{Supp}\nu_{s_i}^2 = S_{-i}$; hence $\mathcal{T}$ is a cautious type structure. For every $i \in I$ and $s_i \in Q_i$, (s_i, s_i) is cautiously rational by construction; for every $s_i' \neq Q_i$, condition (c) of Definition 3 implies that s_i' does not support s_i, so by construction the pair (s_i', s_i) is not rational. Hence, $\mathrm{Proj}_{S_i} R_i^1 = Q_i$. Now, suppose by way of induction that for each $i \in I$ and $s_i \in Q_i$, $(s_i, s_i) \in R_i^m$. We show that type s_i cautiously believes R_{-i}^m, establishing that $(s_i, s_i) \in R_i^{m+1}$; this will yield $(s_i, s_i) \in R_i^\infty$. Note that $\mathrm{Supp}\mu_{s_i}^1 = \{(s_{-i}, s_{-i}) : s_{-i} \in Q_{-i}\} \subseteq R_{-i}^m$, where the inclusion follows from the induction hypothesis. Moreover, since $\mathrm{Supp}\mathrm{marg}\mu_{s_i}^1 = \mathrm{Supp}\nu_{s_i}^1 = Q_{-i}$, Proposition 2.2 entails that R_{-i}^m is cautiously believed under $\beta_i(s_i)$ at level 1. Therefore, we conclude that $\mathrm{Proj}_{S_i} R_i^\infty = Q_i$. ■

Proof of Theorem 3: The desired type structure $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i \in I}$ is constructed as follows. For each $i \in I$, let T_i be the Baire space $\mathbb{N}_0^{\mathbb{N}_0}$,[36] so that each $t_i \in T_i$ is an infinite sequence of non-negative integers. The set $\mathbb{N}_0$ is endowed with the discrete topology, and $\mathbb{N}_0^{\mathbb{N}_0}$ is endowed with the product topology. The basic open sets are sets of the form

$$O_k = \left\{ (n_1, n_2, ...) \in \mathbb{N}_0^{\mathbb{N}_0} \,|(n_1, ..., n_k) = (o_1, ..., o_k) \right\}$$

for each $k \in \mathbb{N}_0$ and $(o_1, ..., o_k) \in (\mathbb{N}_0)^k$. With this topology, a basic open set is also closed, so sets of the form O_k constitute a clopen basis. The space $\mathbb{N}_0^{\mathbb{N}_0}$ is Polish and uncountable, but not compact.

For each $i \in I$, we partition T_i into a countable family of non-empty Borel subsets. For each $k \geq 0$, let

$$\mathrm{T}_i^k = \left\{ (n_1, n_2, ...) \in \mathbb{N}_0^{\mathbb{N}_0} \,|n_1 = k \right\}.$$

[35] Since $\mathrm{Supp}\nu_{s_i}^2 = S_{-i}$, we can also construct $\mu_{s_i}^2$ in such a way that $\mathrm{Supp}\mu_{s_i}^2 = S_{-i} \times T_{-i}$. Then, we would obtain a type structure $\mathcal{T}$ where all types are not only cautious, but also associated with a full-support LPS.

[36] Here $\mathbb{N}_0$ denotes the set $\{0, 1, 2, ...\}$, i.e., $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The Baire space is sometimes defined as the set $\mathbb{N}^{\mathbb{N}}$ of all infinite sequences of natural numbers. This difference is immaterial for all the relevant topological properties we are going to use in this proof.

Each T_i^k is a subbasic clopen subset of T_i; moreover, each T_i^k is homeomorphic to the Baire space. It is clear that $T_i = \cup_{k\geq 0}\mathrm{T}_i^k$, and all the T_i^k's are pairwise disjoint.

The next step is to construct the belief maps in such a way that, for each $k \geq 0$, $t_i \in \mathrm{T}_i^k$ and $s_i \in S_i$, the pair (s_i, t_i) *does not* belong to R_i^{k+1}. For each $i \in I$, we construct a countable partition of $\mathcal{N}(S_{-i} \times T_{-i})$ that "mirrors" the above partition of T_i. This is done as follows: For each $i \in I$, let

$$\Lambda_i^0 = \mathcal{N}(S_{-i} \times T_{-i})\diagdown\mathcal{C}_i^0,$$

where $\mathcal{C}_i^0$ is the set of all LPS's $\overline{\mu}_i \in \mathcal{N}(S_{-i} \times T_{-i})$ such that $\overline{\mathrm{marg}}_{S_{-i}}\overline{\mu}_i \in \mathcal{N}^+(S_{-i})$. Since $|S_i| \geq 2$ for each $i \in I$, it follows that $\Lambda_i^0 \neq \emptyset$.

Next, let

$$\Lambda_i^1 = \left\{\overline{\mu}_i \in \mathcal{C}_i^0 \left| \mu_i^1(S_{-i} \times \mathrm{T}_{-i}^0) > 0 \right.\right\},$$

and, for each $k \geq 2$,

$$\Lambda_i^k = \cap_{m\in\{1,\dots,k-1\}} \left\{\overline{\mu}_i \in \mathcal{C}_i^0 \left| \mu_i^1(S_{-i} \times \mathrm{T}_{-i}^{m-1}) = 0 \right.\right\} \cap \left\{\overline{\mu}_i \in \mathcal{C}_i^0 \left| \mu_i^1(S_{-i} \times \mathrm{T}_{-i}^{k-1}) > 0 \right.\right\}.$$

In words: Λ_i^1 is the set of all LPS's on $S_{-i} \times T_{-i}$ such that the marginal on S_{-i} has full support and the first component measure assigns strictly positive probability to $S_{-i} \times \mathrm{T}_{-i}^0$; Λ_i^2 is the set of all LPS's on $S_{-i} \times T_{-i}$ such that the marginal on S_{-i} has full support and the first component measure assigns probability 0 to $S_{-i} \times \mathrm{T}_{-i}^0$, and strictly positive probability to $S_{-i} \times \mathrm{T}_{-i}^1$; and so on.

It is immediate to check that $\mathcal{N}(S_{-i} \times T_{-i}) = \cup_{k\geq 0}\Lambda_i^k$ and all the Λ_i^k's are non-empty, pairwise disjoint sets; so the countable family of all Λ_i^k's is a partition of $\mathcal{N}(S_{-i} \times T_{-i})$.

Claim C.1 *For each $k \geq 0$, Λ_i^k is a Borel subset of $\mathcal{N}(S_{-i} \times T_{-i})$.*

Proof: Since $\mathcal{C}_i^0$ is Borel (Lemma D.4), so is Λ_i^0. For each $k \geq 1$, let

$$\mathrm{P}_i^k = \left\{\overline{\mu}_i \in \mathcal{C}_i^0 \left| \mu_i^1(S_{-i} \times \mathrm{T}_{-i}^{k-1}) > 0 \right.\right\}.$$

Note that $\Lambda_i^1 = \mathrm{P}_i^1$, and for each $k \geq 2$, Λ_i^k is the intersection of P_i^k with the complements of $\mathrm{P}_i^1, \dots, \mathrm{P}_i^{k-1}$. Thus, to show that each Λ_i^k is Borel in $\mathcal{N}(S_{-i} \times T_{-i})$, it is sufficient to show that each P_i^k is Borel in $\mathcal{N}(S_{-i} \times T_{-i})$. Let

$$\mathrm{M}_i^k = \left\{\mu \in \mathcal{M}(S_{-i} \times \mathrm{T}_{-i}) \left| \mu(S_{-i} \times \mathrm{T}_{-i}^{k-1}) > 0 \right.\right\}.$$

By Theorem 17.24 in Kechris (1995), if X is a Polish space, then the Borel σ-field on $\mathcal{M}(X)$ is generated by sets of the form $\{\mu \in \mathcal{M}(X) : \mu(E) \geq p\}$, where $E \in \Sigma_X$ and $p \in \mathbb{Q} \cap [0,1]$. Hence, for every $E \in \Sigma_X$, the set $\{\mu \in \mathcal{M}(X) : \mu(E) > 0\}$ is Borel, since it can be written as $\cap_{n\in\mathbb{N}} \left\{\mu \in \mathcal{M}(X) : \mu(E) \geq \frac{1}{n}\right\}$. This implies that M_i^k is Borel in $\mathcal{M}(S_{-i} \times \mathrm{T}_{-i})$. Moreover, for each $n \in \mathbb{N}$, the canonical projection map

$$\begin{array}{rccc} \mathrm{Proj}_{1,n}: & \mathcal{N}_n(S_{-i} \times T_{-i}) & \rightarrow & \mathcal{M}(S_{-i} \times \mathrm{T}_{-i}), \\ & (\mu_i^1, \dots, \mu_i^n) & \mapsto & \mu_i^1, \end{array}$$

is continuous, hence the set $\mathrm{Proj}_{1,n}^{-1}\left(\mathrm{M}_i^k\right)$ is Borel in $\mathcal{N}_n(S_{-i} \times T_{-i})$. So, the conclusion follows from the observation that P_i^k can be written as

$$\mathrm{P}_i^k = \left(\cup_{n\in\mathbb{N}}\mathrm{Proj}_{1,n}^{-1}\left(\mathrm{M}_i^k\right)\right) \cap \mathcal{C}_i^0.$$

$\square$

Recall now that every Borel subset of a Polish space is a Lusin space, when endowed with the relative topology. Moreover, every Lusin space is also analytic (see Cohn 2003, Proposition 8.6.13). Thus, by Claim C.1, each Λ_i^k is analytic. Since each T_i^k is homeomorphic to the Baire space, it follows from Corollary 8.2.8 in Cohn (2003; see also Kechris 1995, p. 85) that there exist surjective continuous maps

$$\beta_i^{[k]} : \mathrm{T}_i^k \rightarrow \Lambda_i^k, \forall k \geq 0.$$

For each $i \in I$, let β_i be the union of the $\beta_i^{[k]}$'s, i.e., $\beta_i = \cup_{k \geq 0} \beta_i^{[k]} : T_i \rightarrow \mathcal{N}(S_{-i} \times T_{-i})$. The map is well defined because the T_i^k's are pairwise disjoint. By Fact D.1, β_i is a continuous (and so Borel) surjective map. This completes the definition of the type structure $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i \in I}$.

We now show that $\mathcal{T}$ satisfies the required properties.

Claim C.2 *For each $i \in I$,*

$$(S_i \times \mathrm{T}_i^k) \cap R_i^{k+1} = \emptyset, \forall k \geq 0.$$

Proof: By induction on $k \geq 0$.

(Base step: $k = 0$) Fix $i \in I$ and $(s_i, t_i) \in S_i \times T_i$ with $t_i \in \mathrm{T}_i^0$. We clearly have that $(s_i, t_i) \notin R_i^1$ because $\beta_i(t_i) \in \Lambda_i^0$, hence t_i is not cautious. Therefore $(S_i \times \mathrm{T}_i^0) \cap R_i^1 = \emptyset$.

(Inductive step: $k \geq 1$) Suppose we have already shown that $(S_i \times \mathrm{T}_i^{k-1}) \cap R_i^k = \emptyset$ for each $i \in I$. Fix $i \in I$ and $(s_i, t_i) \in S_i \times T_i$ with $t_i \in \mathrm{T}_i^k$. Thus $\beta_i(t_i) = (\mu_i^1, ..., \mu_i^n) \in \Lambda_i^k$, hence $\mu_i^1(S_{-i} \times \mathrm{T}_{-i}^{k-1}) > 0$. Since, by the induction hypothesis, $(S_{-i} \times \mathrm{T}_{-i}^{k-1}) \cap R_{-i}^k = \emptyset$, it must be the case that $\mu_i^1(R_{-i}^k) < 1$. Therefore R_{-i}^k is not cautiously believed under $\beta_i(t_i)$; this implies $(s_i, t_i) \notin \mathbf{B}_i^c(R_{-i}^k)$. Thus $(s_i, t_i) \notin R_i^{k+1}$. $\square$

To conclude the proof, pick any $(s_i, t_i) \in S_i \times T_i$. Thus there exists $k \geq 0$ such that $t_i \in T_i^k$. By Claim C.2, it follows that $(s_i, t_i) \notin R_i^{k+1}$. Since $R_i^\infty = \cap_{m \geq 0} R_i^m$, this shows that $R_i^\infty = \emptyset$, as required. $\blacksquare$

Appendix D: Measurability of the relevant sets

The aim of this section is to show that, for a given type structure $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i \in I}$, the sets R_i^m, $m > 1$, are Borel subsets of $S_i \times T_i$. We do this by first showing that $\mathbf{B}_i^c(E) \subseteq S_i \times T_i$ is Borel for every event $E \subseteq S_{-i} \times T_{-i}$.

Lemma D.1 *Fix a type structure $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i \in I}$ and non-empty event $E \subseteq S_{-i} \times T_{-i}$. Then the set of all $\overline{\mu} \in \mathcal{N}(S_{-i} \times T_{-i})$ under which E is cautiously believed is Borel in $\mathcal{N}(S_{-i} \times T_{-i})$.*

Proof: Recall that, for a given event $E \subseteq S_{-i} \times T_{-i}$, the set of probability measures μ satisfying $\mu(E) = p$ for $p \in \mathbb{Q} \cap [0, 1]$ is measurable in $\mathcal{M}(S_{-i} \times T_{-i})$. So the sets of all $\mu \in \mathcal{M}(S_{-i} \times T_{-i})$ satisfying $\mu(E) = 1$ or $\mu(E) = 0$ are Borel in $\mathcal{M}(S_{-i} \times T_{-i})$. Now, fix

n and $m \leq n$. By the above argument and by definition of $\mathcal{N}_n(S_{-i} \times T_{-i})$, it turns out that the set

$$\begin{aligned} C^1_{n,m} &= \{\overline{\mu} \in \mathcal{N}_n(S_{-i} \times T_{-i}) \,|\, \mu^m(E) = 1, \forall l \leq m\} \\ &= \cap_{l \leq m} \left\{\overline{\mu} \in \mathcal{N}_n(S_{-i} \times T_{-i}) \,\middle|\, \mu^l(E) = 1\right\} \end{aligned}$$

is Borel in $\mathcal{N}_n(S_{-i} \times T_{-i})$. Note that $C^1_{n,m}$ is the set of all $\overline{\mu} \in \mathcal{N}_n(S_{-i} \times T_{-i})$ for which condition (i) of Definition 12 holds for level m.

By the same argument, it follows that, for every $s_{-i} \in \mathrm{Proj}_{S_{-i}}(E)$, the set

$$\begin{aligned} C^{s_{-i}}_{n.m} &= \left\{\overline{\mu} \in \mathcal{N}_n(S_{-i} \times T_{-i}) \,\middle|\, \mu^l(\{s_{-i}\} \times T_{-i}) = 0, \forall l \leq m\right\} \\ &= \cap_{l \leq m} \left\{\overline{\mu} \in \mathcal{N}_n(S_{-i} \times T_{-i}) \,\middle|\, \mu^l(\{s_{-i}\} \times T_{-i}) = 0\right\} \end{aligned}$$

is Borel in $\mathcal{N}_n(S_{-i} \times T_{-i})$. Note that the set

$$C^2_{n,m} = \bigcap_{s_{-i} \in \mathrm{Proj}_{S_{-i}}(E)} \left(\mathcal{N}_n(\Omega) \diagdown C^{s_{-i}}_{n,m}\right)$$

is the (measurable) set of all $\overline{\mu} \in \mathcal{N}_n(S_{-i} \times T_{-i})$ satisfying condition (ii) of Definition 12 for level m. Define $C_{n,m} = C^1_{n,m} \cap C^2_{n,m}$; clearly, $C_{n,m}$ is a Borel subset of $\mathcal{N}_n(S_{-i} \times T_{-i})$. Hence, the set of all $\overline{\mu} \in \mathcal{N}(S_{-i} \times T_{-i})$ under which E is cautiously believed is given by $\cup_{n \in \mathbb{N}} \cup_{m \in \mathbb{N}} C_{n,m}$, so it is Borel in $\mathcal{N}(S_{-i} \times T_{-i})$. ■

By measurability of each belief map in a lexicographic type structure, we obtain the following result.

Corollary D.1 *Fix a type structure $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i \in I}$. For every $i \in I$, if $E \subseteq S_{-i} \times T_{-i}$ is a non-empty event, then $\mathbf{B}^c_i(E)$ is a Borel subset of $S_i \times T_i$.*

The next step is to show that, for a given type structure $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i \in I}$, the set R^c_i is a Borel subset of $S_i \times T_i$. We first report some auxiliary technical facts we shall be using in the proofs that follow. Fix a countable collection of pairwise disjoint topological spaces $(Y_n)_{n \in \mathbb{N}}$, and let $Y = \cup_{n \in \mathbb{N}} Y_n$. For a given indexed family of mappings $(f_n)_{n \in \mathbb{N}}$, where $f_n : X_n \to Y_n$, let $f : X \to Y$ be the function defined as

$$f(x) = f_n(x), \, x \in X_n.$$

The map $f : X \to Y$ is called the *union* of the functions $(f_n)_{n \in \mathbb{N}}$, and is also denoted by $\cup_{n \in \mathbb{N}} f_n$.

Fact D.1 *Fix two countable collections of pairwise disjoint topological spaces $(X_n)_{n \in \mathbb{N}}$ and $(Y_n)_{n \in \mathbb{N}}$. Let $X = \cup_{n \in \mathbb{N}} X_n$ and $Y = \cup_{n \in \mathbb{N}} Y_n$. Let $(f_n)_{n \in \mathbb{N}}$ be a countable family of mappings $f_n : X_n \to Y_n$. If each map f_n is continuous (resp. Borel measurable), then the union map $\cup_{n \in \mathbb{N}} f_n : X \to Y$ is continuous (resp. Borel measurable).*

Proof: Let O be open in Y. By definition of direct sum topology, the set O can be written as $O = \cup_{n \in \mathbb{N}} O_n$, where each $O_n = O \cap Y_n$ is open in Y_n (see Engelking 1989, p. 74). Thus

$$\left(\cup_{n \in \mathbb{N}} f_n\right)^{-1}(O) = \cup_{n \in \mathbb{N}} f_n^{-1}(O_n).$$

So, if each f_n is continuous (resp. Borel measurable), then each $f_n^{-1}(O_n)$ is open (resp. Borel), and this in turn implies that $(\cup_{n\in\mathbb{N}} f_n)^{-1}(O)$ is open (resp. Borel). ■

Fact D.2 *Let X and Y be Polish spaces, and fix a map $f : X \to Y$. If f is continuous (resp. Borel measurable), then $\widehat{f} : \mathcal{N}(X) \to \mathcal{N}(Y)$ is continuous (resp. Borel measurable).*

Proof: Since $\widehat{f}$ is the union of the functions $\left(\widehat{f}_{(n)}\right)_{n\in\mathbb{N}}$, where $\widehat{f}_{(n)} : \mathcal{N}_n(X) \to \mathcal{N}_n(Y)$, by Fact D.1 it is enough to show that, for each $n \in \mathbb{N}$, $\widehat{f}_{(n)}$ is continuous or Borel measurable. By Theorem 15.14 in Aliprantis and Border (1999), the image measure map $\widetilde{f}$ is continuous provided f is continuous. If f is assumed to be only Borel measurable, we conclude that $\widetilde{f}$ is Borel measurable by using two mathematical facts. First, the Borel σ-field on $\mathcal{M}(X)$ is generated by sets of the form $\{\mu \in \mathcal{M}(X) : \mu(E) \geq p\}$, where $E \in \Sigma_X$ and $p \in \mathbb{Q} \cap [0,1]$. Second, each set $\widetilde{f}^{-1}(\{\nu \in \mathcal{M}(Y) : \nu(E) \geq p\})$, where $E \in \Sigma_Y$, can be written as $\{\mu \in \mathcal{M}(X) : \mu(f^{-1}(E)) \geq p\}$. Since each space $\mathcal{N}_n(X)$ is endowed with the product topology, it follows from Proposition 2.3.6 in Engelking (1989) that the map $\widehat{f}_{(n)}$ is continuous provided f is continuous. The conclusion that $\widehat{f}_{(n)}$ is Borel measurable follows from Lemma 4.49 in Aliprantis and Border (1999). ■

Lemma D.2 *Fix a type structure $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i\in I}$ and $s_i \in S_i$. The set of all $\overline{\mu} \in \mathcal{N}(S_{-i}\times T_{-i})$ such that s_i is a lexicographic best reply to $\overline{\mathrm{marg}}_{S_{-i}}\overline{\mu}$ is Borel in $\mathcal{N}(S_{-i}\times T_{-i})$.*

To prove Lemma D.2, we need the following auxiliary result:

Lemma D.3 *Fix a type structure $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i\in I}$ and $s_i, s_i' \in S_i$. Let $O^{\mathcal{W}}_{s_i,s_i'}$ and $O^{\mathcal{S}}_{s_i,s_i'}$ be subsets of $\mathcal{M}(S_{-i} \times T_{-i})$ defined as follows:*

$$\begin{aligned} O^{\mathcal{W}}_{s_i,s_i'} &= \left\{\mu \in \mathcal{M}(S_{-i} \times T_{-i}) \,\middle|\, \pi_i(s_i, \mathrm{marg}_{S_{-i}}\mu) \geq \pi_i(s_i', \mathrm{marg}_{S_{-i}}\mu)\right\}, \\ O^{\mathcal{S}}_{s_i,s_i'} &= \left\{\mu \in \mathcal{M}(S_{-i} \times T_{-i}) \,\middle|\, \pi_i(s_i, \mathrm{marg}_{S_{-i}}\mu) > \pi_i(s_i', \mathrm{marg}_{S_{-i}}\mu)\right\}. \end{aligned}$$

Then $O^{\mathcal{W}}_{s_i,s_i'}$ and $O^{\mathcal{S}}_{s_i,s_i'}$ are closed and open in $\mathcal{M}(S_{-i} \times T_{-i})$, respectively.

Proof: First recall that the map $\widetilde{\mathrm{Proj}}_{S_{-i}} : \mathcal{M}(S_{-i} \times T_{-i}) \to \mathcal{M}(S_{-i})$ defined by

$$\mu \mapsto \mathrm{marg}_{S_{-i}}\mu,\ \mu \in \mathcal{M}(S_{-i} \times T_{-i}),$$

is continuous. Moreover, for each $\widetilde{s}_i \in S_i$, the function $\pi_i(\widetilde{s}_i, \cdot) : \mathcal{M}(S_{-i}) \to \mathbb{R}$ is also continuous. Define the real valued map $f_{s_i,s_i'} : \mathcal{M}(S_{-i}) \to \mathbb{R}$ as

$$f_{s_i,s_i'}\left(\mathrm{marg}_{S_{-i}}\mu\right) = \pi_i(s_i, \mathrm{marg}_{S_{-i}}\mu) - \pi_i(s_i', \mathrm{marg}_{S_{-i}}\mu),\ \mu \in \mathcal{M}(\Omega).$$

The map $f_{s_i,s_i'}$ is clearly continuous, and the set $O^{\mathcal{W}}_{s_i,s_i'}$ can be written as

$$\begin{aligned} O^{\mathcal{W}}_{s_i,s_i'} &= \left(\widetilde{\mathrm{Proj}}_{S_{-i}}\right)^{-1}\left\{\mathrm{marg}_{S_{-i}}\mu \in \mathcal{M}(S_{-i}) \,\middle|\, f_{s_i,s_i'}\left(\mathrm{marg}_{S_{-i}}\mu\right) \geq 0\right\} \\ &= \left(\widetilde{\mathrm{Proj}}_{S_{-i}}\right)^{-1}\left(f^{-1}_{s_i,s_i'}([0,+\infty))\right) \\ &= \left(f_{s_i,s_i'} \circ \widetilde{\mathrm{Proj}}_{S_{-i}}\right)^{-1}([0,+\infty)), \end{aligned}$$

i.e., $O^{\mathcal{W}}_{s_i,s'_i}$ is the inverse image of the set $[0,+\infty)$, closed in $\mathbb{R}$, under the continuous map $f_{s_i,s'_i} \circ \widetilde{\text{Proj}}_{S_{-i}}$, hence $O^{\mathcal{W}}_{s_i,s'_i}$ is closed in $\mathcal{M}(\Omega)$. An analogous argument shows that set $O^{\mathcal{S}}_{s_i,s'_i}$ can be written as

$$O^{\mathcal{S}}_{s_i,s'_i} = \left(f_{s_i,s'_i} \circ \widetilde{\text{Proj}}_{S_{-i}}\right)^{-1} ((0,+\infty)),$$

hence $O^{\mathcal{S}}_{s_i,s'_i}$ is open in $\mathcal{M}(S_{-i} \times T_{-i})$. ■

Proof of Lemma D.2:[37] Let $U^{s_i}_n$ be the set of all $\overline{\mu} \in \mathcal{N}_n(S_{-i} \times T_{-i})$ for which s_i is a lexicographic best reply to $\overline{\text{marg}}_{S_{-i}}\overline{\mu}$. By Lemma D.3, the sets $O^{\mathcal{W}}_{s_i,s'_i}$ and $O^{\mathcal{S}}_{s_i,s'_i}$ are, respectively, closed and open in $\mathcal{M}(\Omega)$, hence the set $O^{\mathcal{E}}_{s_i,s'_i} = O^{\mathcal{W}}_{s_i,s'_i} \backslash O^{\mathcal{S}}_{s_i,s'_i}$ is closed in $\mathcal{M}(S_{-i} \times T_{-i})$. The set $U^{s_i}_n$ can be expressed as

$$U^{s_i}_n = \bigcap_{s'_i \neq s_i} \left(\begin{array}{c} \left(O^{\mathcal{S}}_{s_i,s'_i} \times \mathcal{N}_{n-1}(\Omega)\right) \cup \left(O^{\mathcal{E}}_{s_i,s'_i} \times O^{\mathcal{S}}_{s_i,s'_i} \times \mathcal{N}_{n-2}(\Omega)\right) \cup \\ ... \cup \left(O^{\mathcal{E}}_{s_i,s'_i} \times O^{\mathcal{E}}_{s_i,s'_i} \times ... \times O^{\mathcal{W}}_{s_i,s'_i}\right) \end{array} \right),$$

and this shows that $U^{s_i}_n$ is Borel in $\mathcal{N}(S_{-i} \times T_{-i})$. The set of all $\overline{\mu} \in \mathcal{N}(S_{-i} \times T_{-i})$ for which s_i is a lexicographic best reply to $\overline{\text{marg}}_{S_{-i}}\overline{\mu}$ can be written as $\cup_{n\in\mathbb{N}} U^{s_i}_n$, hence it is Borel. ■

Given a type structure $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i\in I}$, we let $\mathcal{C}^0_i$ denote the set of all $\overline{\mu} \in \mathcal{N}(S_{-i} \times T_{-i})$ such that $\overline{\text{marg}}_{S_{-i}}\overline{\mu} \in \mathcal{N}^+(S_{-i})$; that is,

$$\mathcal{C}^0_i = \left\{\overline{\mu} \in \mathcal{N}(S_{-i} \times T_{-i}) \left| \overline{\text{marg}}_{S_{-i}}\overline{\mu} \in \mathcal{N}^+(S_{-i}) \right.\right\}.$$

Lemma D.4 *Fix a type structure $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i\in I}$. The set $\mathcal{C}^0_i$ is Borel in $\mathcal{N}(S_{-i} \times T_{-i})$.*

Proof: Note that

$$\mathcal{C}^0_i = \left(\widehat{\text{Proj}}_{S_{-i}}\right)^{-1} \left(\mathcal{N}^+(S_{-i})\right).$$

Since $\mathcal{N}^+(S_{-i})$ is Borel in $\mathcal{N}(S_{-i})$ (see Corollary C.1 in BFK) and the map $\widehat{\text{Proj}}_{S_{-i}} : \mathcal{N}(S_{-i} \times T_{-i}) \to \mathcal{N}(S_{-i})$ is continuous (Fact D.2), it follows that $\mathcal{C}^0_i$ is Borel in $\mathcal{N}(S_{-i} \times T_{-i})$. ■

Corollary D.2 *Fix a type structure $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i\in I}$. For every $i \in I$, the set R^c_i is Borel in $S_i \times T_i$.*

Proof: It follows from Lemma D.4 and from the measurability of β_i that the set $S_i \times \beta_i^{-1}(\mathcal{C}^0_i)$ is Borel in $S_i \times T_i$. Define the set $\overline{R}_i$ as

$$\overline{R}_i = \cup_{s_i \in S_i} \left(\{s_i\} \times \beta_i^{-1}(L^{s_i})\right),$$

where L^{s_i} stands for the set of all $\overline{\mu}_i \in \mathcal{N}(S_{-i} \times T_{-i})$ such that s_i is a lexicographic best reply to $\overline{\text{marg}}_{S_{-i}}\overline{\mu}_i$. By Lemma D.2 and measurability of β_i, it follows that $\overline{R}_i$ is Borel in $S_i \times T_i$. Since $R^c_i = \overline{R}_i \cap \left(S_i \times \beta_i^{-1}(\mathcal{C}^0_i)\right)$, the conclusion follows. ■

[37]The proof closely follows the lines of the proof of Lemma A.6 in Dekel et al. (2016).

We can now state and prove the desired result:

Lemma D.5 *Fix a type structure $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i \in I}$. Then, for each $i \in I$ and each $m > 1$,*

$$R_i^{m+1} = R_i^1 \cap \left(\cap_{l \leq m} \mathbf{B}_i^c \left(R_{-i}^l \right) \right),$$

and R_i^m is Borel in $S_i \times T_i$.

Proof: The equality $R_i^{m+1} = R_i^1 \cap \left(\cap_{l \leq m} \mathbf{B}_i^c \left(R_{-i}^l \right) \right)$ is obvious. By Corollary D.2**,** it follows that, for each $i \in I$, the set $R_i^1 = R_i^c$ is Borel in $S_i \times T_i$. By Corollary D.1, the set $\mathbf{B}_i^c \left(R_{-i}^1 \right)$ is Borel in $S_i \times T_i$. The conclusion follows from an easy induction on m. ■

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