Valence influence in electoral competition with rank objectives*

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November 17, 2017

Abstract

In this paper we examine the effects of valence in a continuous spatial voting model with two incumbent candidates and a potential entrant. All candidates are rank-motivated. We first consider the case where the low valence incumbent (LVC) and the entrant have zero valence, whereas the valence of the high valence incumbent (HVC) is positive. We show that a sufficiently large valence of HVC guarantees a unique equilibrium, where the two incumbents prevent the entry of the third candidate. We also show that an increase in valence allows HVC to adopt a more centrist policy position, while LVC selects a more extreme position. We then consider the non-zero (both positive and negative) valence values of LVC and examine the existence of an equilibrium in that case.

JEL numbers: C72, D72.

Keywords: Valence, Candidates, Electoral Game, Rank Objectives, Electoral Equilibrium, Distribution of Ideal Points.

*The authors wish to acknowledge the support of the Ministry of Education and Science of the Russian Federation, grant #14.U04.31.0002, administered through the NES Center for the Study of Diversity and Social Interactions.
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1 Introduction

In political science, valence usually refers to a feature which is seen in a positive light by all voters (Mueller, 2003, Stokes, 1963). It could relate to experience, trustworthiness and other character traits of parties and candidates. Empirical studies suggest that valence is a key input in voting decisions, sometimes being even more important than policies chosen by parties and candidates (Schofield and Zakharov, 2010, Whiteley et.al., 2005).

Modeling political competition between candidates with a different degree of valence has been a theoretical challenge. No pure-strategy equilibrium exists if candidates are vote-motivated, since the best response of the high-valence candidate (HVC) is to mimic the policy of the low-valence candidate (LVC), which would allow HVC to attract the entire electorate. As a result, the research in this field has shifted to the investigation of mixed-strategy equilibria. The case of two vote-motivated candidates competing in a one-dimensional issue space has been examined by Aragones and Xefteris (2012).[1] They found out that HVC adopts a pure strategy, while LVC mixes two pure strategies equidistant from the left and right of the HVC position. In fact, the positional choice of HVC is independent of the value of the valence advantage, and in the case of a symmetric distribution of voters, it coincides with the ideal point of the median voter. Interestingly, the strategies of the candidates are not affected in the case of sequential platform choices: if HVC moves first it locates near the center of the distribution to make sure that LVC is indifferent between entering to the right and to the left. It is also shown that the distance between the platforms of the two candidates increases with the widening of the valence gap (Berger, Munger, and Pothoff, 2000).[2]

Our analysis is related to a number of previous works, in which no incumbent has a valence advantage. Palfrey (1984) and Weber (1992) considered two incumbents and one entrant, who maximize their share of votes. It was shown (for symmetric voter density in Palfrey (1984) and for any single-peaked density in Weber (1992)) that equilibrium exists and is unique. The entrant always chooses a position between those of the two incumbents, and does not receive a vote share larger than that of any of the incumbents. Those results were later extended in Rubincik and Weber (2007). In Greenberg and Shepsle (1987) there were several incumbents and one entrant. Instead of vote maximization, the entrant's objective is to avoid the last place by obtaining more votes than at least one of the incumbents. This may lead to an entry strategy where the entrant opts for a lower vote share in order to try to reduce the number of votes received by one of the incumbents. The incumbents are often unable to prevent a successful challenger entry, and one can always find a distribution of voters' ideal points that the entry does occur. However, an equilibrium exists in some special cases. For example, Weber (1990) shows the existence of an equilibrium in the discrete case when the number of candidates is not too large. If the voters' ideal points are symmetrically distributed around the mode, in the vote-maximizing model the two incumbents locate closer to the median voter than in the case where the entrant seeks to displace one of the incumbents. The difference between equilibria in the two models depends on the degree to which the distribution of voters' ideal points is concentrated around its mode. Cohen (1987) and Shepsle and Cohen (1990) suggest that if the concentration increases, the difference becomes less significant.

[1] Groseclose (2001) considered candidates who were motivated by both votes and policy.

[2] Similar results for a mixed-strategy equilibrium with two candidates and discrete policy space were obtained by Aragones and Palfrey (2002, 2004) and Hummel (2010). However, in a mixed-strategy equilibrium of the three-candidates setting with a discrete policy space, HVC chooses, on average, a more extreme position than the LVCs (Xefteris, 2014). In Aragones and Palfrey (2005), the policy space is discrete, the position of the median voter is uncertain, and an equilibrium always exists.

In this paper we combine the Palfrey-Weber and Cohen-Shepsle approaches and consider a setting with two incumbents, HVC and LVC, who choose their policy positions simultaneously, and one potential entrant, N. All candidates are rank-motivated and are driven by their place in the electoral competition: being the outright winner is, naturally, the most preferred outcome, followed by a two-way tie for first place, and so forth. The entrant enters the race only if it can achieve at least the sole second place, thereby displacing one of the incumbents.

It is well-known that in a large array of environments involving a quantifiable level of performance, a success is measured by a relative, rather than absolute, performance (see, e.g. Greenberg and Shepsle, 1987). In order to be shortlisted among applicants for a certain job, a candidate should be selected among, say, the top three applicants. In order to qualify for the Champions League, an English, German or Spanish soccer club should guarantee, at least, the forth position in the table, whereas the number of collected points does not matter. The rank objectives are especially prevalent in electoral contests. It is often the case in situations where rank matters that potential entrants are required to surpass the rank of at least one incumbent to be deemed successful. In the first round of presidential elections in many countries, including France, Russia, Poland, Indonesia, and Argentina, candidates must guarantee themselves at least second place in order to advance to the next round. In Britain and Canada, the second largest party has the status of "official opposition" which entitles it to certain perks and privileges. Louisiana holds an "open primary" for governor, with the top two candidates facing each other in a general election thereafter. Under these circumstances, incumbents must consider not only their position relative to the current competition, but also the possibility of displacement by an entrant. A second-place (compared to a third-place) finish in an election significantly increases both rank and vote share in the subsequent election (Anagol and Fujiwara, 2014), perhaps by enabling voters to strategically coordinate their voting decisions on that candidate.

We first consider the case where HVC exhibits a positive degree of valence, while the valence value of LVC and the entrant is zero. We then show that the equilibrium exists if the valence advantage of HVC is large enough. The existence of equilibrium is enforced by a threat of entry by the third candidate. Suppose that the equilibrium position of LVC is to the right of HVC. Then two conditions must be satisfied. First, the support for LVC must be equally divided between voters to the right and to the left of the LVC's position (otherwise, N can claim the larger of these two groups of voters, and relegate LVC to the third place). Second, the share of the vote that N can claim by entering to the left of HVC should not exceed the share of the vote of LVC. It must actually be equal to that share, otherwise, HVC can select a payoff-enhancing position by moving to the right. If the valence advantage is not large enough, the equilibrium unravels, as LVC has a rank-enhancing option of moving to the left. Then N would enter to the left of HVC and squeeze HVC into third place. If the valence advantage is sufficiently large, then HVC is immune to such deviations; instead, LVC will be punished by the entrant, who will choose a position to his right, relegating LVC to third place.

The widening of the valence gap between HVC and LVC always results in HVC choosing a more centrist position, and LVC choosing a more extreme position.[3] Figure 1 shows how the equilibrium policy positions of the two incumbent candidates depend on the magnitude of HVC's valence advantage. On the same graph, we also plot the vote share of HVC.[4]

We then expand our analysis to the case where the valence degree of LVC is different from zero.

[3]Note that these comparative statics are different from Groseclose (2001) in a somewhat different setting.

[4]Here, we assume that the voters' preferences over the policy parameter are single-peaked, with ideal points distributed according to a normal distribution with mean and standard deviation of 0.5, truncated outside interval $[0, 1]$. There are no equilibria if the valence gap is less than approximately 0.095; this threshold value will be different for different distributions of voters' ideal points.

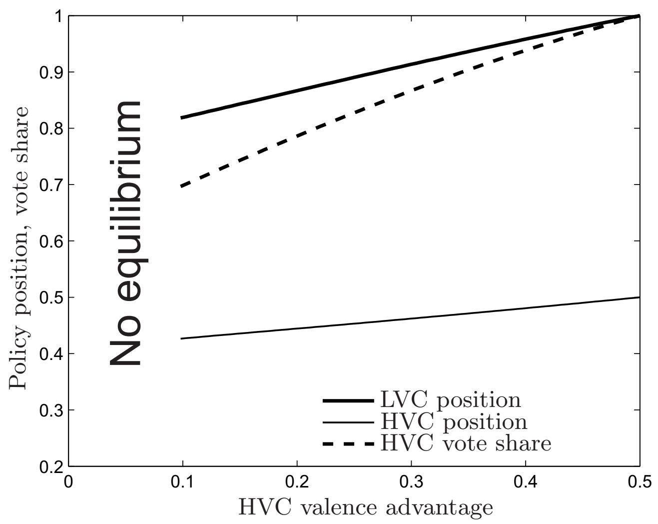

Figure 1: HVC vote share and positions of HVC and LVC, depending on HVC valence advantage under normal distribution of voters' preferences (both mean and standard deviation are 0.5) truncated outside $[0, 1]$. The policy space is $[0, 1]$.

We demonstrate the existence of an equilibrium if LVC's valence is positive but sufficiently small. We then examine the setting where LVC has negative valence, such as resulting from poor track record and experience of the candidate. We derive necessary conditions for the existence of an equilibrium for various classes of distributions of voters' ideal points. Whether the necessary conditions are also sufficient would depend on the assumptions with regard to strategy choices of the entrant. More specifically, in the case where the entrant is indifferent between entering close to HVC and LVC, candidate N has to select the incumbent he wishes to punish by entering in his proximity. It turns out that that selection determines the existence of an equilibrium.

The paper is organized as follows. Section 2 describes the model. Section 3 offers the analysis of the case with a positive degree of valence of HVC, and zero valence of LVC and the entrant. Sections 4 and 5 contain extensions of this setting. Specifically, Section 4 expands the framework of Section 3 to the general case of voters' ideal points distribution and provides the necessary and sufficient conditions for the existence of equilibrium as well as comparative statics analysis. Section 5 examines the case where LVC has a non-zero degree of valence (either positive or negative). Section 6 concludes.

2 The model

There is a continuum of voters. Each voter is characterized by the most preferred alternative, or ideal point in $X = [0, 1]$. Let F be the cumulative distribution of voters' ideal points, with $F(0) = 0$ and $F(1) = 1$. Our first assumption is:

Assumption 1. The probability density $f(x) = F'(x)$ is continuous, differentiable, and strictly positive on X.

The second assumption is known as the *gradually escalating median* (see Haimanko, Le Breton, and Weber, 2005). Roughly speaking, it holds when the density function of voters' ideal points does not have sharp peaks. Formally, take a point $x \in X$ and consider the medians $m_l(x)$ and $m_r(x)$ of the distribution F over intervals $[0, x]$ and $[x, 1]$, respectively. That is, $m_l(x) = F^{-1}\big(F(x)/2\big)$ and $m_r(x) = F^{-1}\big((1 + F(x))/2\big)$ Then if we slightly move x to the right, i.e., consider $x + \epsilon$, where ϵ is a small positive number, the corresponding medians $m_l(x + \epsilon)$ and $m_r(x + \epsilon)$ shift to the right by a distance which is smaller than ϵ.

Assumption 2. For every $x \in X$, we have $m_l'(x) < 1$, $m_r'(x) < 1$.

Denote by

$$\lambda_F = \max_{x,y \in [0,1]} f(x)/f(y),$$

the maximal ratio of values of the density function over all possible pairs $x, y \in [0, 1]$. Since $m_l'(x) = \frac{f(x)}{2f(F(x/2))}$, it is easy to see that the Assumption 2 is satisfied if $\lambda_F < 2$. For our purposes we will also use a slightly tighter bound than is needed in Theorem 2 and several other results.

Assumption 2′. $\lambda_F < \frac{4}{3}$.

There are three candidates, H, L, and N, who compete by choosing policy positions x_H, x_L, x_N from policy space $X = [0, 1]$. Candidates H and L are the high-valence and low-valence incumbent candidates, also labeled HVC and LVC, respectively. Candidate N is the entrant with zero valence, who chooses a position after observing x_H and x_L. He may also decline to participate in the elections, the decision denoted by $x_N = O$.

Given the policy positions chosen by the candidates, each voter picks one of the three candidates. No abstentions are allowed. Voters evaluate the candidates on the basis of their platforms and valence. Define by

$$u_i(y) = \delta_i - |x_i - y|$$

the utility that a voter with the ideal point y would derive from the position chosen by candidate $i = H, L, N$. Let $u_N(y) = -\infty$ if $x_N = O$. The value $\delta_i \geq 0$ is the valence of candidate i. We assume that $\delta_H > \max\{\delta_L, \delta_N\}$. Without loss of generality, we set $\delta_N = 0$, and for the most of the paper, except Section 5, we also let $\delta_L = 0$, denoting $\delta_H = \delta$. Finally, we assume that $\delta \leq \frac{1}{2}$; otherwise HVC can choose the policy position $x_H = \frac{1}{2}$ and win the support of all voters.

A voter with ideal point y is assumed to vote for candidate i for which $u_i(y)$ is the highest. If two candidates deliver the same maximal utility to the voter, then the voter chooses each candidate with probability $\frac{1}{2}$. If all three candidates give the same utility to the voter, he chooses each candidate with probability $\frac{1}{3}$.

Denote by $V_i(x_H, x_L, x_N)$ the vote share of candidate $i = H, L, N$, given positions (x_H, x_L, x_N) of the three candidates. Sometimes we will use the shorthand notation V_i. All candidates have preferences that are lexicographical in rank and vote share. The rank r_i of candidate i is defined as follows:

1. The sole first place,
2. Two-way tie for first place,
3. The sole second place,
4. Three-way tie,
5. Two-way tie for the last place,

6. The sole last place.

We assume that N enters and chooses a position $x_N \in [0,1]$ only if he can attain at least the sole second place.

The goal of this section is to formally introduce a two-person game between the two incumbents, given the subsequent action by the entrant. First, we determine the entrant's decision given (x_H, x_L). Then, we define the expected vote shares of the incumbents given that decision.

For every pair of incumbents' positions (x_H, x_L), let

$$W_1(x_H, x_L) = \{x_N \in [0,1] |\, V_N > \max\{V_H, V_L\}\}$$

be the set of policy positions that guarantee N the sole possession of first place. Similarly, let

$$W_2(x_H, x_L) = \{x_N \in [0,1] |\, V_N = \max\{V_H, V_L\} \text{ and } V_N > \min\{V_H, V_L\}\}$$

be the set of positions that result in a tie for first place between the entrant and one of the incumbents. Also let

$$W_3(x_H, x_L) = \{x_N \in [0,1] |\, V_N < \max\{V_H, V_L\} \text{ and } V_N > \min\{V_H, V_L\}\}$$

be the set of positions that result in the entrant attaining the sole second place.[5] The set of possible entry decisions is then described as follows:

$$E(x_H, x_L) = \begin{cases} W_1(x_H, x_L) & \text{if} \quad W_1(x_H, x_L) \neq \emptyset, \\ W_2(x_H, x_L) & \text{if} \quad W_1(x_H, x_L) = \emptyset, W_2(x_H, x_L) \neq \emptyset, \\ W_3(x_H, x_L) & \text{if} \quad W_1(x_H, x_L) = W_2(x_H, x_L) = \emptyset, W_3(x_H, x_L) \neq \emptyset, \\ \emptyset & \text{if} \quad W_1(x_H, x_L) = W_2(x_H, x_L) = W_3(x_H, x_L) = \emptyset. \end{cases}$$

This set consists of all policy positions x_N that maximize N's rank upon entry, given that the candidate can attain, at least the sole second place. Otherwise, it is empty.

If the set $E(x_H, x_L)$ is empty, N does not enter. The incumbents' vote shares in our two-person game are given by $v_i(x_H, x_L) = V_i(x_H, x_L, O)$ for $i = H, L$. Given x_H and x_L, the best response correspondence of N is defined by

$$b(x_H, x_L) = \begin{cases} \{x_N \in E(x_H, x_L) | \nexists x'_N \in E(x_H, x_L) \text{ such that} \\ V_N(x_H, x_L, x'_N) > V_N(x_H, x_L, x_N)\}, \\ \{O\}, \text{ if } E(x_H, x_L) = \emptyset. \end{cases}$$

That is, a position x_N in $E(x_H, x_L)$ is chosen by N if it can guarantee the highest vote share for N while preserving his maximal possible rank.

The set $b(x_H, x_L)$ could be (i) single-valued; (ii) multi-valued; or (iii) empty.

In case (i), the single best response, denoted by x_N^*, is obviously chosen by N. The incumbents' vote shares are given by $v_i(x_H, x_L) = V_i(x_H, x_L, x_N^*)$ for $i = H, L$.

In case (ii), where $b(x_H, x_L)$ contains more than one point, we will adopt an arbitrary rule $g : [0,1]^2 \to [0,1]$ with the property that, whenever the set $b(x_H, x_L)$ is nonempty and not equal to O, we have $g(x_H, x_L) \in b(x_H, x_L)$. In short, the choice function determines the point of entry for N whenever a best response exists. No restrictions on g will be imposed. Let $\mathcal{G}$ be the set of all such functions g. The incumbents' vote shares are given by $v_i(x_H, x_L) = V_i(x_H, x_L, g(x_H, x_L))$ for $i = H, L$.

[5] In principle, we could have defined W_4, W_5, W_6 and W_7, but we will not utilize this notation in the paper.

Case (iii) is more intricate. Indeed, the best response correspondence can be empty-valued.[6] The reason is that the vote share of N can be discontinuous in x_N at $x_N = x_H - \delta$, $x_N = x_H + \delta$, or at $x_N = x_L$. For example, let voters' ideal points be uniformly distributed, $\delta = 0.1$, $x_H = 0.5$ and $x_L = 0.8$. Then, for $x_N \in (0.3, 0.4)$ we will have $V_N > V_H$ and $V_N > V_L$ with V_N increasing in x_N. But V_N is discontinuous at $x_N = 0.4$, with $\lim_{x \to 0.4^-} V_N(x_H, x_L, x) = 0.4$ and $V_N(x_H, x_L, 0.4) = 0.2$.

However, we still need to define incumbent vote shares for the case where $b(x_H, x_L)$ is empty, but $E(x_H, x_L)$ is not. To address this problem, for every positive ϵ define the set of ϵ-best responses of N to policy positions of HVC and LVC:

$$b_\epsilon(x_H, x_L) = \{x_N \in E(x_H, x_L) | V_N(x_H, x_L, x_N) \geq \sup_{y \in E(x_H, x_L)} V_N(x_H, x_L, y) - \epsilon\}.$$

Consider the set of limit points $b_0(x_H, x_L)$ of the sets $b_\epsilon(x_H, x_L)$ when ϵ approaches zero. It is easy to see that there are two subcases: (iiia) $b_0(x_H, x_L)$ consists of one point, either $x_H \pm \delta$ or x_L; and (iiib) $b_0(x_H, x_L)$ consists of more than one point out of the set $\{x_H - \delta, x_H + \delta, x_L\}$.

(iiia). Suppose that the set of limit points $b_0(x_H, x_L)$ consists of $x_H - \delta$. (The consideration of the cases when this set consists of $x_H + \delta$ or x_L is very similar). We then follow Palfrey's (1984) mechanism. Namely, N randomly chooses a policy position from all positions in $b_\epsilon(x_H, x_L)$, which, if ϵ small enough, consists of an interval $(x', x_H - \delta)$ for some x'. We then evaluate candidates' average vote shares over those sets in the limit, as ϵ tends to zero. Let

$$\mu_\epsilon(x_H, x_L) = \int_{b_\epsilon(x_H, x_L)} dx$$

be the measure of the set of ϵ-best responses. Given ϵ, we can define the expected share of votes for HVC and LVC if the entering N randomly chooses a position that is uniformly distributed on the set of ϵ-best responses. According to Palfrey (1984), the limits of such expected vote shares are well defined for $i = H, L$:

$$v_i(x_H, x_L) = \lim_{\epsilon \to 0} \frac{1}{\mu_\epsilon(x_H, x_L)} \int_{b_\epsilon(x_H, x_L)} V_i(x_H, x_L, x_N) dx_N. \tag{1}$$

(iiib) In this case the set of ϵ-best responses of N, $b_\epsilon(x_H, x_L)$, if ϵ is small enough, consists of disjoint intervals (f.i., $(x', x_H - \delta)$ and (x_L, x'') for some x' and x''). If the choice of the interval by the entrant does not alter the rank order ($r_H < r_L$ or[7] $r_H > r_L$) between HVC and LVC we repeat the Palfrey mechanism and still use definition (1). Otherwise, denote by $b_\epsilon^H(x_H, x_L)$ the subset of the ϵ-best responses for which $r_H < r_L$. Analogously, let $b_\epsilon^L(x_H, x_L)$ be the subset of the ϵ-best responses for which $r_H > r_L$. For example, if the set b_ϵ of the ϵ-best responses consists of two intervals $(x', x_H - \delta)$ and (x_L, x'') for some x' and x'', we put $b_\epsilon^H(x_H, x_L) = (x', x_H - \delta)$ and $b_\epsilon^L(x_H, x_L) = (x_L, x'')$. Then N can minimize the rank of either HVC or LVC, entering closer to him. To resolve the indifference issue, we allow N to make either one of these two choices.

If N prefers to move towards HVC (preserving his own rank unchanged), we consider the subset $b_\epsilon^H(x_H, x_L)$ of the ϵ-best responses. The measure of this set is

$$\mu_\epsilon^H(x_H, x_L) = \int_{b_\epsilon^H(x_H, x_L)} dx.$$

[6]The lack of a well-defined best response function is a common feature of electoral competition models with several incumbents and a potential entrant (Palfrey (1984), Weber (1992, 1997)).

[7]The equality $r_H = r_L$ is impossible under sufficiently small ϵ

Then the corresponding vote shares for $i = H, L$ will be defined as

$$v_i^H(x_H, x_L) = \lim_{\epsilon \to 0} \frac{1}{\mu_\epsilon^H(x_H, x_L)} \int_{b_\epsilon^H(x_H, x_L)} V_i(x_H, x_L, x_N) dx_N. \tag{2}$$

Similarly, if the entrant prefers to move towards LVC, we consider the subset $b_\epsilon^L(x_H, x_L)$ of the ϵ-best responses. The measure of this set will be

$$\mu_\epsilon^L(x_H, x_L) = \int_{b_\epsilon^L(x_H, x_L)} dx.$$

Then the corresponding vote shares for $i = H, L$ will be defined as

$$v_i^L(x_H, x_L) = \lim_{\epsilon \to 0} \frac{1}{\mu_\epsilon^L(x_H, x_L)} \int_{b_\epsilon^L(x_H, x_L)} V_i(x_H, x_L, x_N) dx_N. \tag{3}$$

To summarize, the vote shares defined in this section depend on the way we resolve two types of multiplicity. One is the selection of the choice function g in the case (ii) of multiple best responses. Another is the choice of entry by N in the case (iiib) of indifference between entering close to x_H and x_L. Formally, for every pair J and g, where $J = H, L$ and $g \in \mathcal{G}$, we define the game (J, g), where players' payoffs are given by

$$\tilde{V}_i^{(J,g)}(x_H, x_L) = \begin{cases} V_i(x_H, x_L, g(x_H, x_L)), & \text{if } |b(x_H, x_L)| > 1 \\ v_i^J(x_H, x_L), & \text{if } |b(x_H, x_L)| = 0 \text{ and } |b_0(x_H, x_L)| > 1 \\ v_i(x_H, x_L), & \text{otherwise.} \end{cases} \tag{4}$$

We now consider the electoral competition game between HVC and LVC whose preferences are lexicographic in rank and vote share which was defined by $\tilde{V}_i^{(J,g)}$ in this section. We shall examine the existence of a pure strategy Nash equilibrium and derive comparative statics results. We also demonstrate that the selection of the choice function g in the case of multiple best responses in the case (ii) does not impact our results and our conclusions are invariant with respect to these two choices. The same applies to the choice of entry by N in the case (iiib) of indifference between entering close to x_H and x_L, except the subsection 5.2, where LVC's negative valence ensures that the entrant can displace LVC, breaking the symmetry between the games (L, g) and (H, g).

3 Uniform distribution of voters' ideal points

Consider the case of the uniform distribution of ideal points, which obviously satisfies Assumptions 1, 2, and $2'$. We search for a Nash equilibrium in the electoral competition game with no entry by N. In the following section we show that there is no other equilibrium even in a more general case. Let $\delta_H = \delta \in (0, \frac{1}{2})$ and $\delta_L = \delta_N = 0$.

Let $x_H + \delta < x_L$. Then the ideal point of the voter who is indifferent between HVC and LVC will be given by

$$x_m = \frac{x_H + x_L + \delta}{2}.$$

In any equilibrium, we must have

$$1 - x_L = x_L - x_m, \tag{5}$$

so that the same mass of LVC's voters has ideal points to the left of x_L as to the right of x_L. If this condition is violated with $1 - x_L > x_L - x_m$, then, for $e \in (0, 1 - x_L]$, N can enter with the policy position $x_N = x_L + e$, receiving the share of votes $V_N = 1 - x_L - \frac{e}{2}$ that will be greater than LVC's subsequent vote share $V_L = x_L + \frac{e}{2} - x_m$, if e is sufficiently small. If we have $1 - x_L < x_L - x_m$, then N can enter with policy position $x_N = x_L - e$, $e \in (0, x_L - x_H - \delta)$, obtaining vote share $V_N = \frac{x_L - x_H - \delta}{2}$. That will be greater than LVC's subsequent vote share $V_L = 1 - x_L + \frac{e}{2}$ if e is small enough.

We also must have

$$x_H - \delta = 1 - x_m, \tag{6}$$

so the vote share of LVC must be equal to the mass of voters to the left of $x_H - \delta$. If $x_H - \delta > 1 - x_m = V_L$, then N can enter with $x_N = x_H - \delta - e$, $e \in (0, x_H - \delta]$, and his vote share $V_N = x_H - \delta - \frac{e}{2}$ will exceed V_L if e is small enough. If $x_H - \delta < 1 - x_m = V_L$, then HVC can increase his vote share by choosing $x'_H = x_H + e$, with $e \in (0, x_L - x_H - \delta]$. N will not be able to enter and outrank LVC with any $x_N \in [0, x'_H - \delta)$ if $e > 0$ is sufficiently small.[8] Combining conditions (5) and (6), we get

$$x_H = \frac{2 + \delta}{5} \tag{7}$$

and

$$x_L = \frac{4 + 2\delta}{5}. \tag{8}$$

In an equilibrium, if one exists, the positions of both candidates will shift to the right as δ increases. The vote share of HVC will also increase in δ:

$$V_H = \frac{3 + 4\delta}{5}.$$

If conditions (7) and (8) are satisfied, then one can easily verify that there is no x_N such that N will rank above LVC if he decides to enter.

An additional inequality condition is required for N not to outrank HVC. If $x_N = x_H - \delta - e$ and $e \in (0, x_H - \delta]$, then the share of vote for the entrant will be $V_N = \frac{2-4\delta}{5} - \frac{e}{2}$. It will always be less than $V_L = \frac{2-4\delta}{5}$, and will be smaller than $V_H = x_m - \frac{x_H - \delta + x_N}{2} = \frac{1+8\delta}{5} + \frac{e}{2}$ if

$$\delta \geq \frac{1}{12}. \tag{9}$$

If this condition is satisfied, N will not be able to enter to the left of HVC, relegating him to the third place.

HVC will not deviate from policy position x_H. If he chooses a more rightist position $x'_H > x_H$, he will provoke the entry of N immediately to the left of $x_H - \delta$. Given that, HVC will not benefit from such a shift even if $x'_H \in (x_L - \delta, x_L + \delta)$ and LVC receives zero votes. If HVC shifts left to $x'_H < x_H$, he will lose votes without improving his rank, as N will now enter immediately to the left of x_L.

Likewise, LVC will not change his position: if $x'_L > x_L$, then N would enter to the left of x'_L to gain the second place. Now suppose that LVC deviates to $x'_L = x_m$, and N subsequently enters immediately to the left of $x_H - \delta$. Then N will not outrank HVC if $\frac{x_m + x_H + \delta}{2} - (x_H - \delta) \geq (x_H - \delta)$ or

$$\delta \geq \frac{3}{26}. \tag{10}$$

[8]He *will* be able to enter to the right of LVC's position, pushing him into third place, but that will not reduce HVC's vote share or rank.

In that case, any deviation $x'_L \in [x_m, x_L)$ will instead prompt N to enter to the right of x'_L, relegating LVC to the third place.[9]

We have therefore the following necessary and sufficient conditions for the existence and uniqueness of an equilibrium in the uniform case with no entry of N:

Theorem 1. Assume that the distribution of voters' ideal points is uniform. If $\delta \in [\frac{3}{26}, \frac{1}{2})$, the pair of incumbent strategies $x_H = \frac{2+\delta}{5}$, $x_L = \frac{4+2\delta}{5}$ constitutes a unique no-entry equilibrium. If $\delta < \frac{3}{26}$, an equilibrium with no entry by N fails to exist.

4 General distribution of voters' ideal points

We now turn to the case of a general distribution of voters' ideal points, satisfying our assumptions introduced in Section 2. Let, again, $\delta_H = \delta \in (0, \frac{1}{2})$ and $\delta_L = \delta_N = 0$. First, we establish that no entry occurs in any equilibrium. Then we find necessary and sufficient conditions for equilibrium existence. Finally, we examine the equilibrium comparative statics. All our results of this section hold regardless of the choice of J and g in (4).

4.1 Equilibrium conditions

Our main result shows that, under Assumptions 1 and 2, the equilibrium does not exist if the level of valence δ is sufficiently small.

This leaves open the question of what happens in the case of larger levels of δ. It turns out that, under Assumption $2'$, for any δ in this interval the equilibrium exists and is unique. Moreover, in such an equilibrium the third candidate does not enter.

Theorem 2. Under Assumptions 1 and 2, there exists $\delta_0 \in (0, \frac{1}{2}]$ such that for any $\delta \in (0, \delta_0)$, there are no equilibria in the electoral competition game. However, under Assumptions 1 and $2'$, for any $\delta \in [\delta_0, \frac{1}{2})$ there exists a unique Nash equilibrium. In this equilibrium N does not enter.

In order to prove the second assertion of this theorem we shall show that the necessary and sufficient conditions for a pair (x_H, x_L) to be an equilibrium are as follows:

$$\phi = \gamma = \frac{\alpha}{2}, \tag{11}$$

$$x_H - x_L > \delta, \tag{12}$$

and

$$F\left(\frac{3x_H + x_L + 3\delta}{4}\right) \geq 2\alpha, \tag{13}$$

where $\alpha = F(x_H - \delta)$, $\gamma = F(x_L) - F(\frac{x_H + x_L + \delta}{2})$, and $\phi = 1 - F(x_L)$.

We prove this theorem by establishing a series of intermediate results. Here we only sketch some of the proofs. The formal proofs are relegated to Appendix B. A supplementary result is formulated and proven in Appendix A.

First, we show that in an equilibrium, N is not able to enter and win second place.

[9] We claim that the existence of equilibrium does not depend on our choice of J in the definition of incumbent vote shares (4). This happens because if (x^*_H, x^*_L) are given by (7) and (8), and LVC chooses a different policy position x'_L, then the set of limit points $b_0(x^*_H, x'_L)$ cannot contain multiple elements. Same is true for any deviation x'_H.

Lemma 1. Suppose that (x_H, x_L) is an equilibrium and Assumptions 1 and 2 hold. Then N does not enter.

Assume the contrary and suppose that (x_H, x_L) is an equilibrium in which N enters. In that case, the entrant must have at least the sole second place. Consequently, one of the two incumbents must necessarily rank below second. But that cannot be HVC, as the deviation $x'_H = x_L$ will result in LVC receiving zero votes, and HVC ranking second or above. Therefore, in any entry equilibrium LVC receives the lowest vote share among three candidates.

But it can be shown that, if LVC is ranked below second place, then either HVC or LVC can select a payoff-improving position. In particular, if x_H is far enough to the left, then LVC can choose x'_L such that $1 - F(x'_L) = F(x'_L) - F(\frac{x_H + x'_L + \delta}{2})$, so equal shares of LVC's voters are located to the right and to the left of x'_L. In that case, N will not be able to outrank LVC by choosing a position $x_N \in [x'_L, 1]$. We show that if the distribution of voters' ideal points satisfies Assumption 2 and does not contain sharp spikes, then N will also not be able to obtain a higher vote share than LVC by choosing some $x_N \in (x_H + \delta, x_L)$.

Our next step is to outline the conditions under which a pair of strategies constitutes an equilibrium. For N, the conditions are as follows.

Lemma 2. Let (x_H, x_L) be a pair of strategies for HVC and LVC. Suppose that Assumptions 1 and 2 hold. N does not enter if and only if

$$\phi = \gamma, \tag{14}$$

$$\alpha \leq \min\{\phi + \gamma, \theta + \beta\}, \tag{15}$$

$$\beta \leq \alpha + \theta, \tag{16}$$

where $\alpha = F(x_H - \delta)$, $\theta = F(x_H + \delta) - F(x_H - \delta)$, $\beta = F(\frac{x_H + x_L + \delta}{2}) - F(x_H + \delta)$, $\gamma = F(x_L) - F(\frac{x_H + x_L + \delta}{2})$, $\phi = 1 - F(x_L)$.

Potentially, N can respond in eight ways. He can either choose a position immediately to the left of $x_H - \delta$, immediately to the right of $x_H + \delta$, immediately to the left and right of x_L, positions x_L, $x_H - \delta$, or $x_H + \delta$ themselves, or somewhere in the interval between $x_H + \delta$ and x_L. Conditions (14), (15), and (16) ensure that first seven responses do not provide N with anything better than a tie for the last place. Assumption 2, together with (14) and (16), rules out the last response.

Additional constraints on (x_H, x_L) are imposed by the requirement that HVC and LVC cannot deviate from their strategies.

Lemma 3. Let (x_H, x_L) be such that the conditions of Lemma 2 are satisfied. Then there does not exist an x'_H that gives a higher payoff to HVC, and there does not exist an x'_L that gives a higher payoff to LVC, if and only if

$$\alpha = \gamma + \phi \tag{17}$$

and condition (13) holds.

If condition (17) is violated, then, together with (15), that gives us $\alpha < \gamma + \phi$. Then if N chooses a position immediately to the left of $x_H - \delta$, his vote share will be strictly below $\gamma + \phi$, the vote share of LVC. So, unless $\alpha = \beta + \theta$, HVC will be able to increase his vote share by choosing $x'_H > x_H$, and still prevent the entry of N to his left. If $\alpha = \beta + \theta < \gamma + \phi$, it can also be shown that either HVC or LVC has a payoff-enhancing deviation. Condition (13) makes sure that LVC cannot choose a more leftist policy position such that N will enter with a position to the left of HVC, dropping the latter into third place.

Lemmas 2 and 3 state that conditions $\alpha = \gamma + \phi$, $\gamma = \phi$, $\alpha \leq \theta + \beta$, $\beta \leq \alpha + \theta$, and (13) are necessary and sufficient for an equilibrium to exist. We now need to prove that a pair of candidate strategies that satisfies those conditions exists and is unique if and only if δ is large enough. We proceed with the following statement:

Lemma 4. Let Assumption 1 be satisfied. Then for any $\delta \in (0, \frac{1}{2})$ there exists a solution (x_H^*, x_L^*) to equations $\alpha = \gamma + \phi$ and $\gamma = \phi$, satisfying $x_L^* - x_H^* > \delta$. If Assumption 2 is satisfied, this solution is unique.

One can show that the solution in x_L to equation $\phi = \frac{\alpha}{2}$ is decreasing in x_H. If the distribution of voters' ideal points satisfies Assumption 2, the solution to $\phi = \gamma$ is also increasing in x_H, giving us the unique (x_H^*, x_L^*) that satisfies both conditions of Lemma 4.

As $\frac{3x_H+x_L+3\delta}{4} < \frac{x_H+x_L+\delta}{2}$, the condition (13) is stronger than $\alpha \leq \beta + \theta$. We now show that the condition $\beta \leq \alpha + \theta$ is redundant at (x_H^*, x_L^*):

Lemma 5. Let Assumptions 1 and 2 be satisfied. Then[10] $\beta^* \leq \alpha^*$.

This is a consequeuce of the fact that N always obtains less votes than LVC with any position $x_N \in (x_H + \delta, x_L)$ whenever $\gamma = \phi$ holds, and $\beta = \lim_{x_N \to_+ x_H + \delta_H} V_N$. Finally, we show that the condition (13) is violated if δ is small enough. However, under Assumption $2'$, it is satisfied if and only if valence advantage δ is above a certain threshold. For every $\delta \in (0, \frac{1}{2})$, let

$$D = F\left(\frac{3x_H^* + x_L^* + \delta}{4}\right) - 2\alpha^*.$$

We have the following result:

Lemma 6. Under Assumptions 1 and 2, there exists $\delta_0 \in (0, \frac{1}{2})$ such that $D < 0$ if $\delta < \delta_0$. If Assumption $2'$ also holds, then we also have $D \geq 0$ if $\delta \geq \delta_0$.

To prove this statement, we show that D changes continuously with δ, is positive when δ is close to $\frac{1}{2}$, and negative when δ is close to 0. Assumption $2'$ yields the monotonicity of D with respect to δ.

Theorem 2 then follows from Lemmas 2 - 6.

4.2 Comparative statics

We first examine the impact of a change in the valence level δ on the incumbents' equilibrium positions.

Theorem 3. Let Assumptions 1 and $2'$ be satisfied. Then we have

$$\frac{\partial x_L^*}{\partial \delta} > 0 \text{ and } \frac{\partial x_H^*}{\partial \delta} > 0. \tag{18}$$

[10] We denote by α^* and β^* the values of α and β evaluated at (x_H^*, x_L^*).

As the valence gap widens, LVC will lose votes to his left. He will have to shift his position to the right. This is to guarantee that LVC voter support is split equally between those to the left and to the right of his position. Otherwise, N will be able to enter to one side of LVC or another, and drop him into third place.

An increase in δ will have two countervailing effects on the position of HVC. First, as the vote share of LVC is reduced, N will require a smaller vote share for a successful entry to the left of HVC, who will be forced to shift to the left in order to prevent entry. On the other hand, an increase in the valence of HVC reduces the set of available policy positions to the left of HVC with which N can enter; thus, HVC can gain votes from LVC by shifting his policy position to the right. If the distribution of voters' ideal points satisfies Assumption 2, the second effect dominates, and increase in valence results in HVC adopting a more centrist policy position.

We next look at the effects on equilibrium of changes in the distribution function F. Suppose that a mass a of voters migrates from interval $(x_H^* - \delta, x_m^*)$ to interval $[0, x_H^* - \delta)$, with no movement of voters outside those intervals. We want to see the effect of such a change in voters' preferences on the equilibrium.

Formally, let (x_H^*, x_L^*) satisfy (11) if the distribution of voters' ideal points is $F(x)$. Let $\mathbf{F_a}(x,a)$: $[0,1] \times [0,\infty) \to [0,1]$ be any twice differentiable function such that

1. $\mathbf{F_a}(x,a)$ satisfies Assumptions 1, 2 for all a.
2. $\mathbf{F_a}(x,0) = F(x)$.
3. For all a, $\mathbf{F_a}(x_L^*, a) = F(x_L^*)$, $\mathbf{F_a}(x_m^*, a) = F(x_m^*)$, $\mathbf{f_a}(x_L^*, a) = f(x_L^*)$, and $\mathbf{f_a}(x_m^*, a) = f(x_m^*)$, where $\mathbf{f_a}$ is the derivative of $\mathbf{F_a}$ with respect to x.
4. $\frac{\partial \mathbf{F_a}(x_H - \delta, a)}{\partial a} = 1$ at $a = 0$.

Similarly, define function $\mathbf{F_b}(x,b)$ such that a mass b of voters migrates from (x_m^*, x_L^*) to $(x_L^*, 1]$, with voters outside those intervals remaining stationary. Let (x_{Ha}^*, x_{La}^*) denote the solution to (11) if the distribution of voters' ideal points is $\mathbf{F_a}(x,a)$. Similarly define (x_{Hb}^*, x_{Lb}^*). The following statement is true.

Theorem 4. We have

$$\frac{\partial x_{Ha}^*}{\partial a} < 0, \qquad \frac{\partial x_{Hb}^*}{\partial b} < 0, \tag{19}$$

$$\frac{\partial x_{La}^*}{\partial a} < 0, \qquad \frac{\partial x_{Lb}^*}{\partial b} > 0 \tag{20}$$

at $a = 0$ and $b = 0$.

As the mass of voters to the left of $x_H^* - \delta$ increases, N can make a successful entry with the limit best response $x_N = x_H^* - \delta$. HVC and LVC both shift their positions to the left to prevent this entry.

A migration of voters to the interval $(x_L^*, 1]$ from (x_m^*, x_L^*) has the opposite effect on candidate policy positions. LVC is now threatened with N's limit best response $x_N = x_L^*$, and is forced to shift his position to the right. This move decreases the vote share of LVC, so N now enters to the left of x_H by choosing $x_N = x_H^* - \delta$. To prevent that, HVC must shift his position to the left.

So far we looked at the comparative statics of candidate policy positions with respect to changes in the distribution of voters' ideal points. We will next see how changes in a and b will affect the inequality constraint (13). The following result is available:

Theorem 5. Let Assumptions 1 and 2′ be satisfied. Put

$$\delta_{0a} = \inf\{\delta |\, \text{Condition (13) is satisfied at } (x^*_{Ha}, x^*_{La})\}$$

and

$$\delta_{0b} = \inf\{\delta |\, \text{Condition (13) is satisfied at } (x^*_{Hb}, x^*_{Lb})\}$$

Then

$$\frac{\partial \delta_{0a}}{\partial a} > 0, \ \frac{\partial \delta_{0b}}{\partial b} < 0$$

at $a = 0$ and $b = 0$, respectively.

It turns out that the set of δ for which equilibrium exists shrinks as a increases, and expands as b increases.

The effect of changes in the distribution function $F(\cdot)$ on equilibrium existence can be studied in greater detail using numeric methods. Figure 2 shows the results of a numeric experiment where equilibrium existence was evaluated for various values of δ and various parameters of the distribution function.

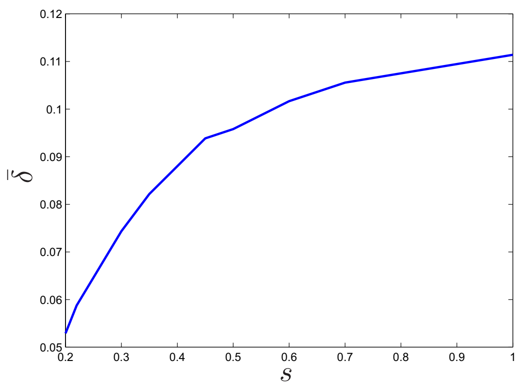

Figure 2: Equilibrium exists if and only if $\delta \geq \bar{\delta}$.

In particular, we assume that voters' ideal points are distributed according to a normal distribution with mean 0.5 and standard deviation s, truncated at $[0, 1]$.[11] The minimum valence advantage for which equilibrium exists increases with s. It turns out that in this example the condition $\beta^* \leq \alpha^*$ of Lemma 5 always holds in equilibrium, even though Assumption 2′ is violated for $s < .66$.

5 General valence values

In this section we expand the analysis of previous sections by removing the requirement that the valence of LVC is equal to zero. Assuming, without loss of generality, that the valence level of the entrant is equal to zero, we consider two cases: the valence of both incumbents is positive, i.e., $0 = \delta_N < \delta_L < \delta_H$ (subsection 5.1) and the negative valence of LVC, when $\delta_L < 0 = \delta_N < \delta_H$ (subsection 5.2).

[11]The experiment was carried out using Matlab 7.11. Equilibrium existence was evaluated for values of δ between 0 and 0.2, in increments of 0.001.

5.1 Positive incumbents' valences

Consider the first case, where $0 = \delta_N < \delta_L < \delta_H$, and assume that the ideal points of the voters are uniformly distributed over $[0,1]$. Note that Lemma 1 in subsection 4.1 can be generalized for the values of $\delta_L \in [0, \delta_H]$, so in no equilibrium N can gain at least the sole second place. If an equilibrium exists, the policy positions of the two incumbent candidates are derived similarly to conditions (7), (8):

$$x_H^* = \frac{2 + \delta_H + 4\delta_L}{5}, \tag{21}$$

$$x_L^* = \frac{4 + 2\delta_H - 7\delta_L}{5}. \tag{22}$$

However, in order for this pair to yield the Nash equilibrium, an additional set of inequalities should be satisfied:

Theorem 6. Under the uniform distribution of voters' ideal points, for a pair of incumbents' levels of valence (δ_H, δ_L), an equilibrium exists if and only if

$$\begin{array}{l} \delta_H \in (\frac{1}{7}, \frac{4}{21}] \text{ and } \delta_L \in (0, \delta_H - \frac{1}{7}], \text{ or} \\ \delta_H \in (\frac{4}{21}, \frac{5}{26}] \text{ and } \delta_L \in (0, 16\delta_H - 3], \text{ or} \\ \delta_H \in (\frac{5}{26}, \frac{1}{2}] \text{ and } \delta_L \in (0, \frac{1-2\delta_H}{8}]. \end{array} \tag{23}$$

Moreover, if those conditions are satisfied, equations (21), (22) determine the Nash equilibrium.

Recall from Theorem 1 that, for $\delta_L = 0$, an equilibrium exists if and only if $\delta_H \in [\frac{3}{26}, \frac{1}{2})$. This exhibits an interesting discontinuity property: for $\delta_H \in [\frac{3}{26}, \frac{1}{7}]$, equilibrium exists if $\delta_L = 0$, but does not exist for any $\delta_L > 0$. This occurs because positive valence allows LVC another way to increase his vote share: he can deviate to $x_L' = x_H^* - \delta_H + \delta_L - e$. If $e \in (0, \delta_L)$, LVC's share of vote will exceed $F(x_H^* - \delta_H)$, which is also equal to LVC's share of vote given (x_H^*, x_L^*). This option is unavailable when $\delta_L = 0$. If $\delta_L > 0$, then (x_H^*, x_L^*) is an equilibrium only if every deviation $x_L' \in (x_H^* - \delta_H, x_H^* - \delta_H + \delta_L)$ is blocked by the subsequent entry of N; this requires additional constraints on δ_H and δ_L.

If we only allow LVC to make deviations $x_L' \in (x_H^* - \delta_H + \delta_L, 1]$ (as well as allowing HVC to make any deviation $x_H' \in [0,1]$), then HVC and LVC will not deviate from positions (x_H^*, x_L^*) if conditions less strict than (23) are met:

$$\begin{array}{l} \delta_H \in (\frac{3}{26}, \frac{5}{26}] \text{ and } \delta_L \in (0, \delta_H - \frac{3}{26}], \text{ or} \\ \delta_H \in (\frac{5}{26}, \frac{1}{4}] \text{ and } \delta_L \in (0, \frac{14\delta_H - 2}{9}], \text{ or} \\ \delta_H \in (\frac{1}{4}, \frac{1}{2}] \text{ and } \delta_L \in (0, \frac{1-2\delta_H}{3}). \end{array} \tag{24}$$

The two sets of conditions (23) and (24) are exhibited on Figure 3.

This figure, drawn for the uniform case, indicates that a generalization of Theorem 6 to an arbitrary distribution of voters' ideal points would be too tedious and complicated. However, a result similar to Theorem 2 can be derived for small values of δ_L:

Theorem 7. Let Assumptions 1 and 2′ be satisfied. Then there is $\delta_H^0 \in (0, \frac{1}{2})$ for which the following property holds: For every $\delta_H \in (\delta_H^0, \frac{1}{2})$ there exists a sufficiently small positive δ_L^0 (depending on δ_H) such that every pair (δ_H, δ_L) with $\delta_L \in [0, \delta_L^0)$ yields a unique Nash equilibrium.

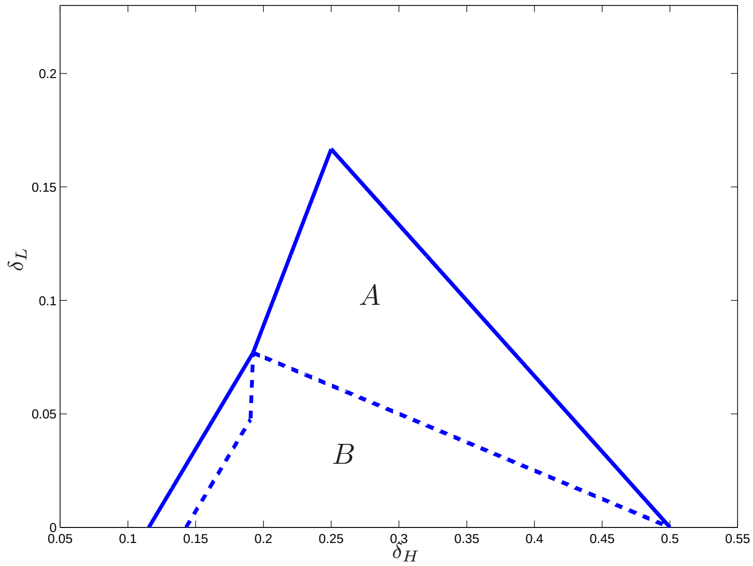

Figure 3: A is the domain of valences (under solid curve) where no deviations $x'_L \in [x_H - \delta_H + \delta_L, 1]$ and no deviations $x'_H \in [0, 1]$ exist if the distribution of voters' preferences is uniform; $B \subset A$ is the set of pairs (δ_H, δ_L) that yields Nash equilibrium.

The proof of this statement follows the lines of the proof of Theorem 2 and we do not present it here. Note only that if an equilibrium exists, the policy positions are derived similarly to (11), to satisfy

$$F(x_L^* + \delta_L) - F(x_m) = 1 - F(x_L^* + \delta_L), \qquad F(x_H^* - \delta_H) = 1 - F(x_m), \tag{25}$$

with $x_m = (x_H^* + \delta_H + x_L^* - \delta_L)/2$. An additional condition

$$F(x_H^* + \delta_H) - F(x_H^* - \delta_H + \delta_L) \geq F(x_H^* - \delta_H + \delta_L) \tag{26}$$

must be satisfied so LVC does not attempt any deviation $x'_L \in (x_H^* - \delta_H, x_H^* - \delta_H + \delta_L)$.[12]

5.2 Negative valence of LVC

Now consider the case where the low-valence incumbent is disadvantaged relative to the entrant: $\delta_L < 0 = \delta_N < \delta_H$. This is the only part of the paper where the outcome depends on the choice of the game (J, g). To recall, in the case of indifference between entering close to x_H and to x_L, N chooses the proximity of LVC (HVC) in the game (L, g) $((H, g))$. The first result of this subsection yields the nonexistence of an equilibrium for a single peaked distribution of voters' ideal points in the game (L, g). We impose

Assumption 3. The probability density $f(x)$ is strictly single-peaked on the interval $X = [0, 1]$.

Theorem 8. Suppose that Assumptions 1 and 3 hold. If $\delta_L < 0 < \delta_H$, $\delta_H - \delta_L < \frac{1}{2}$, then the game (L, g) does not admit an equilibrium.

[12]Taking $\delta_L = 0$, from (26) we obtain δ_H^0, as condition (26) is also sufficient to prevent any deviation $x'_L \in (x_H^* + \delta_H - \delta_L, x_L^*)$.

It turns that an equilibrium does exist for the game (H, g) in the case of a symmetric distribution of voters' ideal points:

Assumption 4. The probability density $f(x)$ is symmetric with regard to $\frac{1}{2}$ on $X = [0, 1]$.

Theorem 9. Suppose that Assumptions 1, 3, and 4 hold. If $\delta_L < 0 < \delta_H$ and $\delta_H - \delta_L < \frac{1}{2}$, then the game (H, g) admits a unique equilibrium. In equilibrium HVC has the highest vote share, followed by N, whereas LVC comes last.

Note that, due to the negative valence of LVC, N can always displace him by choosing a position close to that of LVC. It can also be demonstrated that in any equilibrium, HVC should rank strictly ahead of N. Furthermore, N has three regions for a possible entry: to the left of $x_H - \delta_H$, to the right of $x_L + \delta_L$, and in the interval $(x_H + \delta_H, x_L + \delta_L)$. We shall show that the equilibrium can emerge only if N garners the same vote share under each of the three scenarios. In other words the following conditions should be satisfied:

$$F(x_H - \delta_H) = 1 - F\left(\frac{x_H + x_L + \delta_H + \delta_L}{2}\right) = \sup_{x_N \in (x_H + \delta_H, x_L + \delta_L)} V(N). \tag{27}$$

To provide some intuition for these two results, take, for simplicity, the symmetric case. Consider first the game (H, g). Under (27), candidate N has the limit best response $x_N = x_H - \delta_H$. The incumbents do not possess strategies to increase their payoffs. Indeed, if HVC moves to the left, then N will either have the limit best response $x_N = x_H + \delta_H$ or a best response in the interval $(x_H + \delta_H, x_L + \delta_L)$; in any case, HVC's vote share will be reduced. If HVC moves to the right, N will still have limit best response $x_N = x_H - \delta_H$, and, again, the vote share of HVC will decline. If LVC moves to the left toward HVC, N will have limit best response $x_N = x_L + \delta_L$, completely denying votes to LVC. Finally, if LVC moves to the right, he will lose votes to HVC (or, perhaps, to N entering in the interval $(x_H + \delta_H, x_L + \delta_L)$).

In the game (L, g), the situation is different. N has the limit best response at $x_N = x_L + \delta_L$, with LVC receiving zero votes. Then a slight shift to the right by LVC will result in N switching to $x_N = x_H + \delta_H$, which generates a positive vote share for LVC. Thus, conditions (27) do not yield an equilibrium in this case.

6 Conclusions

We study a model of electoral competition with two incumbents and one entrant. All candidates are driven by their rank in the electoral competition and have various degrees of valence. We first consider the case where LVC and the entrant have zero valence whereas HVC exhibits a positive degree of valence. We then show that there is a unique equilibrium if the valence advantage of HVC is large enough, and that entry is prevented in equilibrium. The candidate positions change as the valence advantage increases: HVC chooses a more centrist position, and the LVC chooses a more extreme position. The minimum value of valence advantage that is required for equilibrium to exist depends on the distribution of voters' ideal points. Numeric analysis suggests that as the variance of voters' ideal points decreases, the emergence of equilibrium is more likely, as it exists for a broader range of δ. The examination of the uniformly distributed voters shows that, except for a small range of δ, the equilibrium is robust to the valence levels' variations. We also cover the case where LVC exhibits either positive or negative degree of valence, that could be explained by the previous (successful or not) record of the candidate. We show that for a small positive degree of LVC's valence the equilibrium may still exist. In the case of the negative valence we derive a set of necessary conditions for the existence of an equilibrium.

The basic assumption of the paper is that the valence values are determined exogenously. It would be very interesting and challenging to examine the case where valence levels depend on the history of candidates. However, it would require, at least, a two-stage model, which is left for future research.

Appendix A. Supplementary result.

The following result will be used in the proof of Lemmas 1, 2, and 3.

Lemma 7. Let Assumptions 1 and 2 be satisfied. Take (x_H, x_L) such that $x_H + \delta < x_L < 1$. If $1 - F(x_L) \geq F(x_L) - F(\frac{x_H+x_L+\delta}{2})$, then there does not exist $x_N \in (x_H + \delta, x_L)$ such that $V_N > V_L$. If $F(\frac{x_H+x_L+\delta}{2}) \leq 2F(x_H + \delta)$, then there does not exist $x_N \in (x_H + \delta, x_L)$ such that $V_N > V_H$.

Proof of Lemma 7. Under Assumption 2, we have $m_l'(x) < 1$ and $m_r'(x) < 1$. Let $1 - F(x_L) \geq F(x_L) - F(\frac{x_H+x_L+\delta}{2})$ and $x_N \in (x_H + \delta, x_L)$. We have $m_r(\frac{x_H+x_L+\delta}{2}) \geq x_L$. That gives us

$$
\begin{aligned}
m_r\left(\frac{x_H+x_N+\delta}{2}\right) &= m_r\left(\frac{x_H+x_L+\delta}{2}\right) - \int_{\frac{x_H+x_N+\delta}{2}}^{\frac{x_H+x_L+\delta}{2}} m_r'(x)dx > \\
> \; m_r\left(\frac{x_H+x_L+\delta}{2}\right) &- \frac{x_H+x_L+\delta}{2} + \frac{x_H+x_N+\delta}{2} \geq \\
= \; x_L - \frac{x_H+x_L+\delta}{2} &+ \frac{x_H+x_N+\delta}{2} = \frac{x_N+x_L}{2}.
\end{aligned}
$$

By definition,

$$
1 - F\left(m_r\left(\frac{x_H+x_N+\delta}{2}\right)\right) = F\left(m_r\left(\frac{x_H+x_N+\delta}{2}\right)\right) - F\left(\frac{x_H+x_N+\delta}{2}\right),
$$

so

$$
1 - F\left(\frac{x_N+x_L}{2}\right) > F\left(\frac{x_N+x_L}{2}\right) - F\left(\frac{x_H+x_N+\delta}{2}\right)
$$

and $V_L > V_N$. Let $F(\frac{x_H+x_L+\delta}{2}) \leq 2F(x_H+\delta)$ and $x_N \in (x_H+\delta, x_L)$. We have $m_l(\frac{x_H+x_L+\delta}{2}) \leq x_H+\delta$. That gives us

$$
\begin{aligned}
m_l\left(\frac{x_N+x_L}{2}\right) &= m_l\left(\frac{x_H+x_L+\delta}{2}\right) + \int_{\frac{x_H+x_L+\delta}{2}}^{\frac{x_N+x_L}{2}} m_l'(x)dx < \\
< \; m_l\left(\frac{x_H+x_L+\delta}{2}\right) &+ \frac{x_N+x_L}{2} - \frac{x_H+x_L+\delta}{2} \leq \\
< \; x_H + \delta + \frac{x_N+x_L}{2} &- \frac{x_H+x_L+\delta}{2} = \frac{x_H+x_N+\delta}{2}.
\end{aligned}
$$

By definition,

$$
F\left(\frac{x_N+x_L}{2}\right) - F\left(m_l\left(\frac{x_N+x_L}{2}\right)\right) = F\left(m_l\left(\frac{x_N+x_L}{2}\right)\right), \text{ so}
$$

$$
F\left(\frac{x_N+x_L}{2}\right) - F\left(\frac{x_H+x_L+\delta}{2}\right) < F\left(\frac{x_H+x_L+\delta}{2}\right),
$$

and $V_H > V_N$. $\square$

Appendix B. Proofs of statements.

Proof of Lemma 1. In this and subsequent proofs, we will drop the (J, g) notation from $\tilde{V}_i^{(J,g)}$, as all our statements apply to all $J = H, L$ and $g \in \mathcal{G}$. Denote $\tilde{V}_N = 1 - \tilde{V}_H - \tilde{V}_L$ to be the expected vote share of candidate N.

Suppose that (x_H, x_L) is an equilibrium that does not prevent the entry of candidate N. Then we must have $\tilde{V}_N > \tilde{V}_L$ or $\tilde{V}_N > \tilde{V}_H$. HVC cannot rank below sole second place in equilibrium, because deviation $x'_H = x_L$ will guarantee him at least second place, with candidate L ranking last with zero vote share. Therefore we have $\tilde{V}_H > \tilde{V}_L$ and $\tilde{V}_N > \tilde{V}_L$. We will show that candidate H or L always has a payoff-improving deviation.

Case 1. $\theta < \min\{\alpha, \alpha'\}$, where $\alpha = F(x_H - \delta)$ and $\alpha' = 1 - \alpha - \theta$.

Case 1A: $\alpha \leq \alpha'$. Take $x'_L = x_H - \delta - e$, where $e > 0$. Denote by $\tilde{V}'_i = \tilde{V}_i(x_H, x'_L)$, for $i = H, L, N$. If e is small enough, the limit best response (attained from the right) will be $x_N = x_H + \delta$, with $\tilde{V}'_L > \tilde{V}'_H$, which is a contradiction.

Case 1B: $\alpha > \alpha'$. This case is symmetric to Case 1A.

Case 2: $\theta \geq \min\{\alpha, \alpha'\}$ and $\alpha < \alpha'$. Let $\hat{x}_L$ be such that $1 - F(\hat{x}_L) = F(\hat{x}_L) - F(\frac{x_H + \hat{x}_L + \delta}{2})$.

Case 2A: $1 - F(\hat{x}_L) \geq \frac{\alpha}{2}$.

Case 2A1: $\hat{x}_L \neq x_L$. Take $x'_L = \hat{x}_L$, and denote $V'_i = V_i(x_H, x'_L, x_N)$. If $x_H > \delta$ and $x_N \in [0, x_H - \delta)$, we will have $V'_N < \alpha \leq V'_L$. If $x_H \geq \delta$ and $x_N = x_H - \delta$, we have $V'_N = \frac{\alpha}{2} < V'_L$. If $x_N \in (x_H - \delta, x_H + \delta)$, then $V'_N = 0$. If $x_N \in (x_H + \delta, \hat{x}_L)$, we have $V'_L > V'_N$ by Lemma 7.

If $x_N = x_H + \delta$, then $V'_N = \frac{1}{2}(F(\hat{x}_L) - F(\frac{x_H + \hat{x}_L + \delta}{2}))$. If $x_N = \hat{x}_L$, then $V'_N = V'_L$ because $1 - F(\hat{x}_L) = F(\hat{x}_L) - F(\frac{x_H + \hat{x}_L + \delta}{2})$. If $x_N \in (\hat{x}_L, 1]$, then $V'_N \leq V'_L$. Thus, for all $x_N \in [0, 1]$, we have $V_L \geq V_N$, so $\tilde{V}_N \leq \tilde{V}_L$. Candidate L improves her rank with $x'_L = x_L$, and (x_H, x_L) is not an equilibrium.

Case 2A2: $\hat{x}_L = x_L$. By an argument identical to Case 2A1, we have $b(x_H, x_L) = O$, which contradicts our assumption.

Case 2B: $1 - F(\hat{x}_L) < \alpha/2$. Consider the following cases.

Case 2B1: $x_L < x_H - \delta$. Then the limit best response will be $x_N = x_H + \delta$. It follows that for any $x'_L \in (x_L, x_H - \delta)$, the limit best response will also be $x'_N = x_H + \delta$, and we will have $\tilde{V}_L < \tilde{V}'_L < \tilde{V}_N$, with rank of candidate L remaining the same. So, (x_H, x_L) is not an equilibrium.

Case 2B2: $x_L \in (\bar{x}_L, 1]$, where $\bar{x}_L$ is the solution to $F(\frac{x_H + x_L + \delta}{2}) = 1 - \alpha$. Then $\tilde{V}_L < \alpha$, with candidate L ranking last by assumption that (x_H, x_L) is an equilibrium. If $x'_L = x_H - \delta - e$, then the limit best response will be $x_N = x_H + \delta$, and we will have $\tilde{V}'_L > \tilde{V}_L$ if e is small enough. So, (x_H, x_L) is not an equilibrium.

Case 2B3: $x_L \in (x_H + \delta, \bar{x}_L)$. Because $\alpha/2 > 1 - F(\hat{x}_L)$, we have $x_L < \hat{x}_L$ and $1 - F(x_L) > F(x_L) - F(\frac{x_H + x_L + \delta}{2})$. Take $x'_H = x_H + e$. If e is small enough, we will have $1 - F(x_L) > F(x_L) - F(\frac{x'_H + x_L + \delta}{2})$.

Because of this and $\theta \geq \alpha$, by Lemma 7 we will have $V'_L > V'_N$ and $V'_H > V'_N$ for all $x_N \in (x_H + \delta, x_L)$. The limit best response will be $x_N = x_L$ for both x_H and x'_H. Hence $\tilde{V}_H(x'_H, x_L) > \tilde{V}_H(x_H, x_L)$, with rank of candidate H remaining the same. So, (x_H, x_L) is not an equilibrium.

Case 2B4: $x_L = \bar{x}_L$. Candidate N has limit best response $x_N = x_L$ (argument is similar to one for Case 2B3). Let $x'_L = \bar{x}_L + e < \hat{x}_L$. Then candidate N will have limit best response $x_N = x_H - \delta$ (attained from the left). We will have $\tilde{V}'_L > \tilde{V}_L$, with candidate L ranking last in either case.

Case 3: $\alpha' \leq \alpha$ and $\theta \geq \min\{\alpha, \alpha'\}$ This case is symmetric to Case 2.

□

Proof of Lemma 2. Conditions (14) and (15) imply that $x_H + \delta < x_L < 1$. Indeed, if $x_H + \delta > x_L$, then $\gamma < 0$, and (14) is violated, as $\phi \geq 0$. If $x_H + \delta = x_L$, then $\gamma = \phi = 0$, so $x_L = 1$ and $\alpha > 0$, as, by assumption, we have $\delta < \frac{1}{2}$. This contradicts (15). As $x_H + \delta < x_L$, we must have $\beta > 0$ and $\gamma = \phi > 0$, so $x_L < 1$.

We are first going to show that, if the conditions (14), (15), and (16) are satisfied, then there does not exist $x_N \in [0, 1]$ such that $V_N > V_H$ or $V_N > V_L$. As a consequence, there does not exist a best response $x_N \in [0, 1]$ or a limit best response for candidate N. Indeed, if $x_N \in [0, x_H - \delta)$, then we have $V_N = F(\frac{x_H + x_N - \delta}{2}) < \alpha \leq \theta + \beta < V_H$ and $V_N < \gamma + \phi = V_L$. If $x_N = x_H - \delta$, then $V_N = \frac{\alpha}{2} < V_H = \frac{\alpha}{2} + \theta + \beta$ and $V_N < V_L = \gamma + \phi$. If $x_N \in (x_H - \delta, x_H + \delta)$, then $V_N = 0$. If $x_N = x_H + \delta$, then $V_N = \frac{\beta}{2} < V_H = \frac{\beta}{2} + \theta + \alpha$ and $V_N < V_L = \gamma + \phi$. If $x_N \in (x_H + \delta, x_L)$, then, as $\gamma = \phi$ and $\beta \leq \alpha + \theta$, by Lemma 7 we have $V_L > V_N$ and $V_H > V_N$. Suppose that $x_N = x_L$. We have $V_L = V_N = \gamma$ and $V_H = \alpha + \theta + \beta$. We have $V_H \geq V_N$ because $\alpha + \theta + \beta = \lim_{x \to x_L^-} V_H(x_H, x_L, x)$ and $\gamma = \lim_{x \to x_L^-} V_N(x_H, x_L, x)$, but $V_H(x_H, x_L, x_N) > V_N(x_H, x_L, x_N)$ for all $x_N \in (x_H + \delta, x_L)$. Finally, if $x_N \in (x_L, 1]$, then we have $V_N = 1 - F(\frac{x_L + x_N}{2}) < \gamma \leq \alpha + \theta + \beta = V_H$ and $V_N < V_L$, as $V_N + V_L = 2\gamma$.

We will now show that if conditions (14), (15), and (16) are violated, there will exist $x_N \in [0, 1]$ such that $V_N > V_H$ or $V_N > V_L$, that trigger the entry by candidate N. Let $x_H + \delta < x_L$. If $\gamma < \phi$, we will have $V_N > V_L$ if $x_N = x_L + e$ and $e > 0$ is small enough. If $\gamma > \phi$, we will have $V_N > V_L$ if $x_N = x_L - e$ and $e > 0$ is small enough. If $\alpha > \gamma + \phi$, we will have $V_N > V_L$ if $x_N = x_H - \delta - e$ and $e > 0$ is small enough. If $\alpha > \theta + \beta$, we will have $V_N > V_H$ if $x_N = x_H - \delta - e$ and $e > 0$ is small enough. Finally, if $\beta > \theta + \alpha$, we will have $V_N > V_H$ if $x_N = x_H + \delta + e$ and $e > 0$ is small enough.

Let $x_H + \delta = x_L < 1$. Then $V_N > V_L$ if $x_N = x_H + \delta + e$ and $e > 0$ is small enough. If either $x_H + \delta = x_L = 1$ or $x_H + \delta > x_L$, then $V_L = 0$ for all $x_N \in [0, 1]$, and there exists $e > 0$ such that $V_N > 0$ if either $x_N = x_H - \delta - e$ or $x_N = x_H + \delta + e$. □

Proof of Lemma 3. Let the conditions of Lemma 2 be satisfied. Then, clearly, $b(x_H, x_L) = O$. Suppose that $\alpha < \gamma + \phi$ and $\alpha = \beta + \theta$. Consider the following cases.

Case 1: $\phi < \alpha$. Let $x_L' = x_L - e$. If e is small enough, candidate N will have limit best response $x_N = x_H - \delta$, with $\tilde{V}_L' = 1 - F(\frac{x_L' + x_H + \delta}{2}) > 1 - F(\frac{x_L + x_H + \delta}{2}) = \tilde{V}_L$, $\tilde{V}_H' < \beta + \theta < \gamma + \phi < \tilde{V}_L'$, and $\alpha = \tilde{V}_N' < \tilde{V}_L'$, so candidate L improves his payoff.

Case 2: $\phi = \alpha$. Then we have $\tilde{V}_H = \tilde{V}_L$. Let $x_H' = x_H - e$. Then candidate N has limit best response $x_N = x_L$. We have $\tilde{V}_H' < \tilde{V}_H$, but, if e is small enough, we will have $\tilde{V}_H' > \tilde{V}_N'$ and $\tilde{V}_H' > \tilde{V}_L'$, so candidate H will improve his rank.

Case 3: $\phi > \alpha$. Let $x_H' = x_H + e$. Then candidate N has limit best response $x_N = x_L$. We have $\tilde{V}_H' = F(\frac{x_L + x_H' + \delta}{2}) > F(\frac{x_L + x_H + \delta}{2}) = \tilde{V}_H$. Recall from the proof of Lemma 1 that $\alpha + \beta + \theta \geq \gamma = \phi$. By that we will have $\tilde{V}_H' > \tilde{V}_N'$ and $\tilde{V}_H' > \tilde{V}_L'$.

It follows that either candidate H or candidate L will be able to improve his payoff.

Suppose that $\alpha < \gamma + \phi$ and $\alpha < \beta + \theta$. Let $x_H' = x_H + e$. Similarly to Case 3 above, candidate H will be able to improve his payoff.

It follows that, if the conditions of Lemma 2 are satisfied, condition $\alpha = \gamma + \phi$ is necessary for no payoff-improving deviations for candidates H and L to exist. Suppose that this condition is satisfied. We will show that candidate H has no payoff-improving deviations, and candidate L has a payoff-improving deviation if and only if condition (13) is violated.

Note that candidate H is ranked first and consider $x_H' \neq x_H$. If $x_H' \in [0, x_H)$, we will have $\tilde{V}_H' < \tilde{V}_H$, because $\tilde{V}_H' \leq F(\frac{x_H' + x_L + \delta}{2})$. If $x_H' \in (x_H, 1]$, then candidate N will have limit best response

$x_N = \min\{x_H - \delta, x_L\}$. This gives us $\tilde{V}_N' \geq \alpha$, $\tilde{V}_L' < \alpha$, and $\tilde{V}_H' \leq 1 - \alpha = \tilde{V}_H$. So, there is no x_H' such that $\tilde{V}_H' > \tilde{V}_H$.

Consider $x_L' \neq x_L$:

Case 1: $x_L' \in [0, x_H - \delta)$. We have $\tilde{V}_L' < \alpha = \gamma + \phi = \tilde{V}_L$. If candidate N has limit best response $x_N = x_H + \delta$, we will have $\tilde{V}_L' < \tilde{V}_N$, as $\alpha < \beta + \gamma + \phi$. If $b(x_H, x_L') = O$, then $\tilde{V}_L < \theta + \beta + \gamma + \phi < \tilde{V}_H$, so candidate L will not improve his rank by deviating to x_L', but will reduce his share of vote.

Case 2: $x_L' = x_H - \delta$. We have $\tilde{V}_L' = \frac{\alpha}{2} < \tilde{V}_L$, with candidate L not ranking higher than second for same reasons as with $x_L' \in [0, x_H - \delta)$.

Case 3: $x_L' \in (x_H - \delta, x_H + \delta)$. $\tilde{V}_L = 0$.

Case 4: $x_L' \in [x_H + \delta, x_m)$: the limit best response of candidate N will be $x_N = x_L'$, with $\tilde{V}_N > \tilde{V}_L$ and $\tilde{V}_H > \tilde{V}_L$.

Case 5: $x_L' \in (x_m, x_L)$. Put

$$x_m' = \frac{x_H + x_L' + \delta}{2}.$$

Take $x_N \in [0, x_H - \delta]$. Then $V_N' < \alpha$. We have $\lim_{x \to (x_H - \delta)^-} V_N(x_H, x_L', x) = \alpha = \gamma + \phi < \lim_{x \to (x_H - \delta)^-} V_L(x_H, x_L', x)$ and $\lim_{x \to (x_H - \delta)^-} V_H(x_H, x_L', x) = F(x_m') - \alpha$. If $x_N \in (x_H - \delta, x_H + \delta)$, then $V_N' = 0$. As $\alpha + \theta \geq \beta > F(x_m') - F(x_H + \delta)$, by Lemma 7 we have $V_N' < V_H'$ whenever $x_N \in (x_H + \delta, x_L')$. We also have $V_N' < V_H'$ when $x_N = x_H + \delta$. If $x_N = x_L'$, then $V_N' = V_L' = \frac{1 - F(x_m')}{2}$.

As $x_L = m_r(x_m)$, we have

$$m_r(x_m') = x_L - \int_{x_m'}^{x_m} m_r'(x)dx > \frac{x_L + x_L'}{2} > x_L',$$

so $F(x_m') - F(x_L') < 1 - F(x_L')$. Thus, by Lemma 7, we have $V_N' < V_L'$ whenever $x_N \in (x_H + \delta, x_L')$. We have $\lim_{x \to x_L'^+} V_N(x_H, x_L', x) = 1 - F(x_L') > F(x_m') - F(x_L') = \lim_{x \to x_L'^+} V_L(x_H, x_L', x)$ and

$$\lim_{x \to x_L'^+} V_H(x_H, x_L', x) = V_H(x_H, x_L', x_L').$$

As $\alpha = \gamma + \phi$, we have $\lim_{x \to x_L'^+} V_N(x_H, x_L', x) < \lim_{x \to (x_H - \delta)^-} V_N(x_H, x_L', x)$. Limit best response of candidate N will be as follows:

1. If $\alpha > F(x_m') - \alpha$, candidate N will have limit best response $x_N = x_H - \delta$,
2. If $\alpha \leq F(x_m') - \alpha$, candidate N will have limit best response $x_N = x_L'$.

Candidate L will not deviate to $x_L' \in (x_m, x_L)$ if and only if $2\alpha \leq F(x_m')$. This will be true for all $x_L' \in (x_m, x_L)$ if and only if condition (13) holds.

Case 6: $x_L' = x_m$. Consider a limiting case $x_N \to_- x_H - \delta$. Then we have $V_N = \alpha$, $V_H = F(x_m') - \alpha$, and $V_L = 1 - F(x_m')$. The ranking of the candidates will be as follows. If $F(x_m') > 2\alpha$, then $V_L > V_N$ and $V_H > V_N$. If $F(x_m') = 2\alpha$, then $V_L > V_H = V_N$. If $F(x_m') < 2\alpha$, then $V_L > V_N > V_H$. In another limiting case, $x_N \to_+ x_L'$, we have $V_N = \alpha$, $V_H = F(x_m')$, and $V_L = 1 - F(x_m') - \alpha$, with $V_H > V_N > V_L$. It follows that candidate N will have limit best response $x_N = x_L'$ if $F(x_m') \geq 2\alpha$.

Case 7: $x_L' \in (x_L, 1]$. We will have $\tilde{V}_L' < \tilde{V}_L$. As $\gamma = \phi$, for $x_N = x_L - e$ we will have $V_N' > V_L'$ if e is small enough. So, candidate N will enter. We will also have $\tilde{V}_L' < \tilde{V}_H$ and $\tilde{V}_L' < \tilde{V}_N'$.

It follows that, when $\alpha = \gamma + \phi$, candidate L will not be able to improve its payoff if and only if (13) holds.

□

Proof of Lemma 4. Denote $f_L = f(x_L)$, $f_{H-} = f(x_H - \delta)$, and $f_m = f(x_m)$.

Define

$$H_1 = 1 + F\left(\frac{x_H + x_L + \delta}{2}\right) - 2F(x_L) \tag{28}$$

and

$$H_2 = F(x_H - \delta) + 2F(x_L) - 2. \tag{29}$$

Conditions (11) then become $H_1 = 0$ and $H_2 = 0$. We have the following derivatives:

$$\begin{array}{ll} \frac{\partial H_1}{\partial \delta} = \frac{1}{2} f_m & \frac{\partial H_2}{\partial \delta} = -f_{H-} \\ \frac{\partial H_1}{\partial x_H} = \frac{1}{2} f_m & \frac{\partial H_2}{\partial x_H} = f_{H-} \\ \frac{\partial H_1}{\partial x_L} = \frac{1}{2} f_m - 2f_L & \frac{\partial H_2}{\partial x_L} = 2f_L \end{array} \tag{30}$$

Let $x_L^2(x_H)$ be a solution to $H_2 = 0$. Since $f(x) > 0$ for all $x \in [0, 1]$, by the Implicit Function Theorem we have

$$\frac{\partial x_L^2(x_H)}{\partial x_H} = -\frac{f_{H-}}{2f_L} < 0. \tag{31}$$

We have $x_L^2(\delta) = 1$ and, since $\frac{\partial H_2}{\partial x_L} < 0$, we must have $x_L^2(1-\delta) < 1$. As $H_1(\delta, 1) = F(\frac{1}{2} + \delta) - 1 < 0$ and $H_1(1 - \delta, x_L^2(1 - \delta)) = 1 + F(\frac{1 + x_L^2(1-\delta)}{2}) - 2F(x_L^2(1 - \delta)) > 0$, for some $x_H \in (\delta, 1 - \delta)$ we have $H_1(x_H, x_L^2(x_H)) = 0$. It follows that system (11) has a solution.

Now let Assumption 2 be satisfied. Let $x_L^1(x_H)$ be a solution to $H_1 = 0$. Since $f(x) > 0$ for all $x \in [0, 1]$, by the Implicit Function Theorem we have

$$\frac{\partial x_L^1(x_H)}{\partial x_H} = -\frac{\frac{1}{2} f_m}{\frac{1}{2} f_m - 2f_L} > 0, \tag{32}$$

since from Assumption 2 it follows that $2f_L > \frac{1}{2} f_m$. Therefore there exists only one x_H such that $x_L^1(x_H) = x_L^2(x_H)$. It follows that system (11) has a unique solution.

□

Proof of Lemma 5. As $\gamma^* = \phi^*$, we have

$$1 - F(x_L^*) = F(x_L^*) - F\left(\frac{x_H^* + x_L^* + \delta}{2}\right).$$

We have $\lim_{x_N \to + x_H^* + \delta_H} V_N = \beta^*$ and $\lim_{x_N \to + x_H^* + \delta_H} V_L = \alpha^*$. Hence, by Lemma 7, we must have $\beta^* \leq \alpha^*$. □

Proof of Lemma 6. Denote $f_{H+} = f(x_H + \delta)$. Let

$$M = \begin{pmatrix} \frac{\partial H_1}{\partial x_H} & \frac{\partial H_1}{\partial x_L} \\ \frac{\partial H_2}{\partial x_H} & \frac{\partial H_2}{\partial x_L} \end{pmatrix} \tag{33}$$

be the Jacobian matrix of $(H_1 H_2)$. We have

$$|M| = f_m f_L - f_{H-}\left(\frac{1}{2} f_m - 2f_L\right), \tag{34}$$

and a sufficient condition for $|M| > 0$ is $4f_L \geq f_m$, which is satisfied because of Assumption $2'$. The derivatives of x_H^* and x_L^* with respect to δ will be given by the the Implicit Function Theorem:

$$\begin{pmatrix} \frac{\partial x_H^*}{\partial \delta} \\ \frac{\partial x_L^*}{\partial \delta} \end{pmatrix} = -\frac{1}{|M|} \begin{pmatrix} 2f_L & 2f_L - \frac{1}{2}f_m \\ -f_{H-} & \frac{1}{2}f_m \end{pmatrix} \begin{pmatrix} \frac{1}{2}f_m \\ -f_{H-} \end{pmatrix},$$

with all densities evaluated at (x_H^*, x_L^*). It follows that

$$\begin{aligned} \frac{\partial x_H^*}{\partial \delta} &= -\frac{1}{|M|}(f_L(f_m - 2f_{H-}) + \frac{1}{2}f_m f_{H-}), \\ \frac{\partial x_L^*}{\partial \delta} &= \frac{1}{|M|} f_m f_{H-}. \end{aligned} \tag{35}$$

Denote $f_t = f((3x_H + x_L + 3\delta)/4)$. By the Implicit Function Theorem and Assumption 2, D varies continuously with δ:

$$\begin{aligned} \frac{\partial D(x_H^*, x_L^*)}{\partial \delta} &= \frac{1}{4}\left(3\frac{\partial x_H^*}{\partial \delta} + \frac{\partial x_L^*}{\partial \delta} + 3\right) f_t - 2\left(\frac{\partial x_H^*}{\partial \delta} + 1\right) f_{H-} = \\ &= \frac{1}{|M|}\left(f_{H-}f_m\left(4f_L - \frac{f_t}{4}\right) + f_t f_L\left(2f_{H-} - \frac{f_m}{2}\right)\right). \end{aligned} \tag{36}$$

If Assumption $2'$ is also satisfied, this derivative is positive. As $D < 0$ for δ close to 0 and $D > 0$ for δ close to $\frac{1}{2}$, we have $D < 0$ if $\delta < \delta_0$ and $D > 0$ if $\delta > \delta_0$ for some $\delta_0 \in (0, \frac{1}{2})$ under Assumption $2'$.
□

Proof of Theorem 3. The second of the derivatives (35) is positive. Now suppose that $f_L = af_m$ and $f_{H-} = bf_m$. We have $\frac{\partial x_H^*}{\partial \delta} = -f_m^2/|M| \cdot (a(1-2b) + \frac{b}{2})$. We must find the minimum $c < 1$ for which inequality $(a(1-2b) + \frac{b}{2}) \geq 0$ will hold for all a, b such that $a \in [c, 1/c]$, $b \in [c, 1/c]$, and $a/b \in [c, 1/c]$. The solution to this problem is $c = \frac{3}{4}$. Hence, the derivative is positive as long as Assumption $2'$ is true. □

Proof of Theorem 4. Assuming distribution function $\mathbf{F_a}(x, a)$ and differentiating (28) and (29) at $a = 0$, we get

$$\frac{\partial H_1}{\partial a} = 0, \ \frac{\partial H_2}{\partial a} = 1. \tag{37}$$

Similarly, at $b = 0$ we have

$$\frac{\partial H_1}{\partial b} = 2, \ \frac{\partial H_2}{\partial b} = -2. \tag{38}$$

By the Implicit Function Theorem, (30) and (33), at $a = b = 0$ we have

$$\begin{pmatrix} \frac{\partial x_H}{\partial a} & \frac{\partial x_L}{\partial a} \\ \frac{\partial x_H}{\partial b} & \frac{\partial x_L}{\partial b} \end{pmatrix} = -M^{-1} \cdot \begin{pmatrix} \frac{\partial H_1}{\partial a} & \frac{\partial H_1}{\partial b} \\ \frac{\partial H_2}{\partial a} & \frac{\partial H_2}{\partial b} \end{pmatrix} = \begin{pmatrix} f_m - 4f_L & -2f_m \\ -f_m & 4f_{H-} + 2f_m \end{pmatrix} \cdot \frac{1}{2|M|},$$

where $|M|$ is given by (34), and all densities are evaluated at (x_H^*, x_L^*). The required signs of partial derivatives follow from Assumption 2. □

Proof of Theorem 5. Assuming distribution functions $\mathbf{F_a}(x,a)$ and $\mathbf{F_b}(x,b)$, we differentiate D with respect to a and b:

$$\frac{\partial D}{\partial a} = -2,\ \frac{\partial D}{\partial b} = 0. \tag{39}$$

Let

$$\bar{M} = \begin{pmatrix} \frac{\partial H_1}{\partial x_H} & \frac{\partial H_1}{\partial x_L} & \frac{\partial H_1}{\partial \delta} \\ \frac{\partial H_2}{\partial x_H} & \frac{\partial H_2}{\partial x_L} & \frac{\partial H_2}{\partial \delta} \\ \frac{\partial D}{\partial x_H} & \frac{\partial D}{\partial x_L} & \frac{\partial D}{\partial \delta} \end{pmatrix}, \tag{40}$$

where

$$\frac{\partial D}{\partial x_H} = \frac{3}{4}f_t - 2f_{H-},\ \frac{\partial D}{\partial x_L} = \frac{1}{4}f_t,\ \text{and}\ \frac{\partial D}{\partial \delta} = \frac{1}{4}f_t + 2f_{H-}. \tag{41}$$

Taking $D = 0$, at $a = b = 0$ we have

$$\begin{pmatrix} \frac{\partial x^*_{Ha}}{\partial a} & \frac{\partial x^*_{Hb}}{\partial b} \\ \frac{\partial x^*_{La}}{\partial a} & \frac{\partial x^*_{Lb}}{\partial b} \\ \frac{\partial \delta_a}{\partial a} & \frac{\partial \delta_b}{\partial b} \end{pmatrix} = -\bar{M}^{-1} \cdot \begin{pmatrix} \frac{\partial H_1}{\partial a} & \frac{\partial H_1}{\partial b} \\ \frac{\partial H_2}{\partial a} & \frac{\partial H_2}{\partial b} \\ \frac{\partial D}{\partial a} & \frac{\partial D}{\partial b} \end{pmatrix} \tag{42}$$

where $\frac{\partial D}{\partial \delta}$ is evaluated keeping x_H, x_L constant. Evaluating this expression, we get

$$\frac{\partial \delta_a}{\partial a} = (2f_m f_L - \frac{1}{4}f_m f_t + \frac{3}{2}f_L f_t)\frac{1}{|M|} \tag{43}$$

$$\frac{\partial \delta_b}{\partial b} = (-2f_m f_{H-} + \frac{1}{2}f_m f_t - \frac{1}{2}f_{H-} f_t)\frac{1}{|M|}, \tag{44}$$

where

$$|\bar{M}| = 4f_{H-}f_m f_L - \frac{1}{4}f_{H-}f_m f_t + 2f_{H-}f_L f_t - \frac{1}{2}f_m f_L f_t,$$

and all densities are evaluated at (x^*_H, x^*_L). If Assumption $2'$ is satisfied, then all derivatives have the required signs. □

Proof of Theorem 6. Suppose that candidates have strategies (x^*_H, x^*_L) defined by (21), (22), and candidate H has no payoff-improving deviations. Candidate L has positive vote share if $x^*_H + \delta_H < x^*_L + \delta_L$, or

$$\delta_L < \frac{1 - 2\delta_H}{3}. \tag{45}$$

Candidate N should not be able to outrank candidate H with any x_N to the left of $x^*_H - \delta_H$. The corresponding condition, derived similarly to (9), is

$$\delta_H - \delta_L \geq \frac{1}{12}. \tag{46}$$

We now derive conditions (24). Candidate L cannot make a deviation $x'_L \in (x^*_H + \delta_H - \delta_L, x_L)$ if after such a deviation candidate N can enter with x_N to the right of $x'_L + \delta_L$. Candidate N will not follow through with such a move only if he can instead enter with x_N to the left of $x^*_H - \delta_H$, pushing candidate H to third place, and, at the same time, getting a higher share of the votes than with any x_N to the right of $x'_L + \delta_L$. To derive a condition similar to (10) and (13) that disallows such deviations by candidate L we need to consider two cases.

*C*ase 1. If $x_H^* \geq \frac{1}{2}$ or

$$\delta_L \geq \frac{1-2\delta_H}{8}, \tag{47}$$

then, for any $x'_L \in (x_H^* + \delta_H - \delta_L, x_L)$, candidate N can choose a position to the left of $x_H^* - \delta_H$ that will give him a higher vote share than any position to the right of $x'_L + \delta_L$. Hence, no payoff-improving deviation for Candidate L will exist if and only if candidate N cannot rank above candidate H for any x_N to the left of $x_H^* - \delta_H$ and for any $x'_L \in (x_H^* + \delta_H - \delta_L, x_L^*)$. That amounts to $x_H^* - \delta_H \leq 2\delta_H - \delta_L$, or

$$\delta_L \leq \frac{14\delta_H - 2}{9}. \tag{48}$$

If this condition holds, any attempt by candidate L to move closer to candidate H will result in candidate N entering to the right of candidate L, pushing him to the third place.

*C*ase 2. If $x_H^* < \frac{1}{2}$ or

$$\delta_L < \frac{1-2\delta_H}{8}, \tag{49}$$

then for all $x'_L \in (x_H^* + \delta_H - \delta_L, \frac{x_H^*+\delta_H+x_L^*-3\delta_L}{2})$, candidate N can obtain a higher vote share if he enters to the right of $x'_L + \delta_L$ rather than to the left of $x_H^* - \delta_H$. It follows that in equilibrium, candidate N should not be able to outrank candidate H if $x'_L = \frac{x_H^*+\delta_H+x_L^*-3\delta_L}{2}$, or

$$\delta_H - \delta_L \geq \frac{3}{26}. \tag{50}$$

As candidate L's vote share will be zero whenever $x_L \in (x_H - \delta_H + \delta_L, x_H + \delta_H - \delta_L)$, there will be no payoff-improving deviation $x'_L \in [x_H^* - \delta_H + \delta_L, 1]$ if either (47) and (48), or (49) and (50), are satisfied. Together with (45) and the excessive (46) this gives us conditions (24).

Suppose now that conditions (24) are satisfied. Our goal is to derive the conditions on δ_H and δ_L under which candidate L cannot improve his payoff with a deviation $x'_L \in [0, x_H - \delta_H + \delta_L)$.

Before deriving the conditions, we will introduce some notation for vote shares of the three candidates, depending on x'_L and the possible response of candidate N. Let

$$\bar{V}_L = \tilde{V}_L(x_H^*, x_L^*) = 1 - \frac{x_H^* + x_L^* + \delta_H - \delta_L}{2} = \frac{2 - 4\delta_H + 4\delta_L}{5}$$

be candidate L's vote share with (21), (22). For $x'_L \in [0, x_H^* - \delta_H + \delta_L)$, let

$$\hat{V}_N = \lim_{x_N \to_+ x_H^* + \delta_H} V_N(x'_L, x_H^*, x_N) = \frac{3 - 6\delta_H - 4\delta_L}{5},$$

$$\hat{V}_L(x'_L) = \lim_{x_N \to_+ x_H^* + \delta_H} V_L(x'_L, x_H^*, x_N) = \frac{x'_L}{2} + \frac{2 - 4\delta_H + 9\delta_L}{10},$$

$$\hat{V}_H(x'_L) = \lim_{x_N \to_+ x_H^* + \delta_H} V_H(x'_L, x_H^*, x_N) = \frac{2 + 16\delta_H - \delta_L}{10} - \frac{x'_L}{2}.$$

These are the limits of the vote shares of the three candidates if candidate N chooses a position immediately to the right of $x_H^* + \delta_H$. Let

$$V_{L0} = \hat{V}_L(0) = \frac{2 - 4\delta_H + 9\delta_L}{10} \text{ and } V_{L1} = \lim_{x'_L \to x_H^* - \delta_H + \delta_L} \hat{V}_L(x'_L) = \frac{2 - 4\delta_H + 9\delta_L}{5}$$

be the smallest and largest (in the limit) vote shares that candidate L can obtain with $x'_L \in [0, x^*_H - \delta_H + \delta_L)$, given such response from candidate N. Similarly define

$$V_{H0} = \hat{V}_H(0) = \frac{2 + 16\delta_H - \delta_L}{10} \text{ and } V_{H1} = \lim_{x'_L \to x^*_H - \delta_H + \delta_L} \hat{V}_H(x'_L) = 2\delta_H - \delta_L.$$

Now suppose that, whenever $x'_L \in [0, x^*_H - \delta_H - \delta_L]$, candidate N chooses a position immediately to the right of $x'_L + \delta_L$, and does not enter when $x'_L \geq x^*_H - \delta_H - \delta_L$. Define

$$V'_L(x'_L) = \begin{cases} \lim_{x_N \to_+ x'_L + \delta_L} V_L(x'_L, x^*_H, x_N), & x'_L \in [0, x^*_H - \delta_H - \delta_L] \\ V_L(x'_L, x^*_H, O), & x'_L \in (x^*_H - \delta_H - \delta_L, x^*_H - \delta_H + \delta_L), \end{cases}$$

or

$$V'_L(x'_L) = \begin{cases} \delta_L + x'_L, & x'_L \in [0, \frac{2-4\delta_H-\delta_L}{5}] \\ \frac{2-4\delta_H+9\delta_L}{10} + \frac{x'_L}{2}, & x'_L \in (\frac{2-4\delta_H-\delta_L}{5}, \frac{2-4\delta_H+9\delta_L}{5}). \end{cases}$$

Similarly, let

$$V'_N(x'_L) = \begin{cases} \frac{2-4\delta_H-\delta_L}{10} - \frac{x'_L}{2}, & x'_L \in [0, \frac{2-4\delta_H-\delta_L}{5}] \\ 0, & x'_L \in (\frac{2-4\delta_H-\delta_L}{5}, \frac{2-4\delta_H+9\delta_L}{5}). \end{cases}$$

Note that the vote share of candidate H must be greater than one half whenever $x'_L \in [0, x^*_H - \delta_H + \delta_L)$ and $x_N < x^*_H - \delta_H$ (or $x_N = O$). This is true because

$$x^*_H - \delta_H + \delta_L = \frac{2 - 4\delta_H + 9\delta_L}{5} < \frac{1}{2} \text{ whenever } \delta_L < \frac{1 + 8\delta_H}{18},$$

which must hold if (24) are satisfied. Let

$$x' = \frac{2 - 4\delta_H - 11\delta_L}{15}$$

be such that $V'_L(x') = V'_N(x')$. Note that $V'_L(x'_L) < V'_N(x'_L)$ if $x'_L \in [0, x')$, and $V'_L(x'_L) > V'_N(x'_L)$ if $x'_L \in (x', x^*_H - \delta_H - \delta_L)$.

Finally, suppose that candidate N chooses a position immediately to the left of $x'_L - \delta_L$ whenever $x'_L > \delta_L$, and $x_N = O$ otherwise. Define

$$V''_L(x'_L) = \begin{cases} V_L(x^*_H, x'_L, O), & x'_L \in [0, \delta_L] \\ \lim_{x_N \to_- x_L - \delta_L} V_L(x^*_H, x'_L, x_N) & x'_L \in (\delta_L, x^*_H - \delta_H + \delta_L) \end{cases}$$

or

$$V''_L(x'_L) = \begin{cases} \frac{2-4\delta_H+9\delta_L}{10} + \frac{x'_L}{2}, & x'_L \in [0, \delta_L] \\ \frac{2-4\delta_H+19\delta_L}{10} - \frac{x'_L}{2}, & x'_L \in (\delta_L, \frac{2-4\delta_H+9\delta_L}{5}). \end{cases}$$

Similarly, let

$$V''_N(x'_L) = \begin{cases} 0, & x'_L \in [0, \delta_L] \\ x'_L - \delta_L, & x'_L \in (\delta_L, \frac{2-4\delta_H+9\delta_L}{5}). \end{cases}$$

Let

$$x'' = \frac{2 - 4\delta_H + 29\delta_L}{15}$$

be such that $V''_L(x'') = V''_N(x'')$. Note that $V''_L(x'_L) > V''_N(x'_L)$ if $x'_L \in [\delta_L, x'')$, and $V''_L(x'_L) < V''_N(x'_L)$ if $x'_L \in (x'', x_H - \delta_H + \delta_L)$. We also have $x'' > x'$.

Note that $V_{H0} > V_{L0}$ as $\delta_L < 2\delta_H$ in any equilibrium. Similarly, we always have $\bar{V}_L > V_{L0}$ as $\delta_L < 2 - 4\delta_H$, and we should always have $\bar{V}_L < V_{L1}$. As $\hat{V}_H(x'_L)$ is decreasing in x'_L and $\hat{V}_L(x'_L)$ is increasing in x'_L, we have $\hat{V}_N > V_{L1} \Rightarrow \hat{V}_N > V_{L0}$ and $\hat{V}_N > V_{H0} \Rightarrow \hat{V}_N > V_{H1}$.

Any (δ_H, δ_L) satisfying conditions (24) falls into one of the nine cases (see Figure 4):

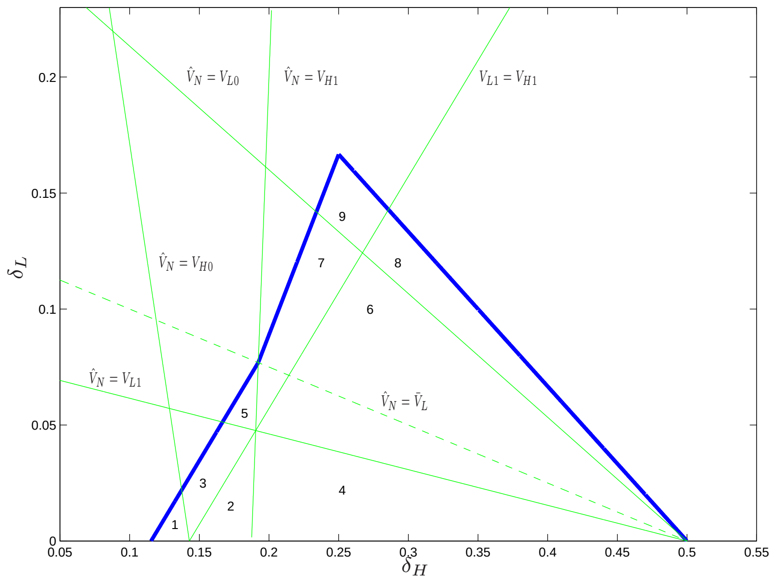

Figure 4: Equilibrium conditions (24).

Case 1. $\hat{V}_N \geq V_{L1}$ and $\hat{V}_N > V_{H0}$, or

$$\delta_L \leq \frac{1 - 2\delta_H}{13} \text{ and } \delta_L < \frac{4 - 28\delta_H}{7}. \tag{51}$$

There **exists** $x'_L \in [0, x^*_H - \delta_H + \delta_L)$ that improves the payoff of candidate L. Take $x'_L = x^*_H - \delta_H + \delta_L - e$. Candidate N will have the limit best response $x'_N = x^*_H + \delta_H$, ranking first. If $e > 0$ is sufficiently small, we will have $\hat{V}_L(x'_L) > \hat{V}_H(x'_L)$. This will be true because condition $\hat{V}_N > V_{H0}$ (which can be rewritten as $\delta_H < \frac{1}{7} - \frac{\delta_L}{4}$) implies $V_{L1} > V_{H1}$ (or $\delta_H < \frac{1}{7} - \delta_L$). We will also have $\hat{V}_L(x'_L) > \bar{V}_L$ if e is sufficiently small, because $\hat{V}_{L1} > \bar{V}_L$. So, candidate L will improve his vote share without changing his rank.

Case 2. $V_{H1} \geq V_{L1}$ and $\hat{V}_N > V_{H1}$, or

$$\delta_L \leq \delta_H - \frac{1}{7} \text{ and } \delta_L > \max\{0, 16\delta_H - 3\}.$$

There **does not exist** $x'_L \in [0, x^*_H - \delta_H + \delta_L)$ that improves the payoff of candidate L. For any $x'_L \in [0, x^*_H - \delta_H + \delta_L)$, candidate N will have the limit best response $x'_N = x^*_H + \delta_H$. Let $\bar{x}$ be such that $\hat{V}_N(\bar{x}) = \hat{V}_H(\bar{x})$. Then, for all $x'_L \in [0, \bar{x})$, we will have $\hat{V}_H(x'_L) \geq \hat{V}_N > \hat{V}_L(x'_L)$, and for any $x'_L \in (\bar{x}, x^*_H - \delta_H + \delta_L)$, we will have $\hat{V}_N > \hat{V}_H(x'_L) > \hat{V}_L(x'_L)$. So, the rank of candidate L will decrease for any $x'_L \in [0, x^*_H - \delta_H + \delta_L)$.

Case 3. $\hat{V}_N \geq V_{L1}$, $V_{H1} < V_{L1}$, and $\hat{V}_N \leq V_{H0}$, or

$$\delta_L \leq \frac{1-2\delta_H}{13}, \ \delta_L > \delta_H - \frac{1}{7}, \text{ and } \delta_L \geq \frac{4-28\delta_H}{7}.$$

There **exists** $x'_L \in [0, x^*_H - \delta_H + \delta_L)$ that improves the payoff of candidate L. Take $x'_L = x^*_H - \delta_H + \delta_L - e$. If $e > 0$ is sufficiently small, then candidate N will have the limit best response $x'_N = x^*_H + \delta_H$, with $\hat{V}_N > \hat{V}_L(x'_L) > \hat{V}_H(x'_L)$ and $\hat{V}_L(x'_L) > \max\{\bar{V}_L, \hat{V}_H(x'_L)\}$. So, candidate L will be better off.

Case 4. $V_{H1} \geq \hat{V}_N$ and $\hat{V}_N \geq V_{L1}$, or

$$\delta_L \leq 16\delta_H - 3 \text{ and } \delta_L \leq \frac{1-2\delta_H}{13}.$$

There **does not exist** $x'_L \in [0, x^*_H - \delta_H + \delta_L)$ that improves the payoff of candidate L. For any such x'_L, candidate N will have the limit best response $x'_N = x^*_H + \delta_H$, so $\hat{V}_H(x'_L) > \hat{V}_N > \hat{V}_L(x'_L)$, and the rank of candidate L will decrease.

Case 5. $V_{H1} < \hat{V}_N$ and $\hat{V}_N < V_{L1}$, or

$$\delta_L > 16\delta_H - 3 \text{ and } \delta_L > \frac{1-2\delta_H}{13}.$$

There **exists** $x'_L \in [0, x^*_H - \delta_H + \delta_L)$ that improves the payoff of candidate L. We have $\delta_L < \frac{1-2\delta_H}{8}$, so $x^*_H < \frac{1}{2}$. Take $x'_L = x^*_H - \delta_H + \delta_L - e$. If $e > 0$ is sufficiently small, we have $V''_N(x'_L) \approx x^*_H - \delta_H < \hat{V}_N$. It follows that candidate N has the limit best response $x'_N = x^*_H + \delta_H$ that yields $\hat{V}_L(x'_L) > \hat{V}_N > \hat{V}_H(x'_L)$, so candidate L will be better off.

Case 6. $\hat{V}_N < V_{L1}$, $V_{H1} \geq V_{L1}$, and $\hat{V}_N \geq V_{L0}$, or

$$\delta_L > \frac{1-2\delta_H}{13}, \ \delta_L \leq \delta_H - \frac{1}{7} \text{ and } \delta_L \leq \frac{1-2\delta_H}{4.25}.$$

There **exists** $x'_L \in [0, x^*_H - \delta_H + \delta_L)$ that improves the payoff of candidate L **if and only if** $\delta_L > \frac{1-2\delta_H}{8}$. Indeed, such x'_L must satisfy the following conditions:

1. $\hat{V}_L(x'_L) > \bar{V}_L$, or $x'_L \in (\frac{2-4\delta_H-\delta_L}{5}, x^*_H - \delta_H + \delta_L)$.
2. $\hat{V}_L(x'_L) \geq \hat{V}_N$, or $x'_L \in [\frac{4-8\delta_H-17\delta_L}{5}, x^*_H - \delta_H + \delta_L)$. Otherwise, there exists $x'_N > x^*_H + \delta_H$ such that $V_H(x'_L, x^*_H, x'_N) > V_N(x'_L, x^*_H, x'_N) > V_L(x'_L, x^*_H, x'_N)$.
3. $V'_L(x'_L) \geq V'_N(x_L)$, or $x'_L \in (x', x^*_H - \delta_H + \delta_L)$. Otherwise, there exists $x'_N > x'_L + \delta_L$ such that $V_H(x'_L, x^*_H, x'_N) > V_N(x'_L, x^*_H, x'_N) > V_L(x'_L, x^*_H, x'_N)$.
4. $V''_L(x'_L) \geq V''_N(x_L)$, or $x'_L \in [0, x'']$. Otherwise, there exists $x'_N < x'_L - \delta_L$ such that $V_H(x'_L, x^*_H, x'_N) > V_N(x'_L, x^*_H, x'_N) > V_L(x'_L, x^*_H, x'_N)$.

We have $x' < x''$. It is true that $x'' < \frac{2-4\delta_H-\delta_L}{5} < \frac{4-8\delta_H-17\delta_L}{5}$ if $\delta_L < \frac{1-2\delta_H}{8}$, $x'' > \frac{2-4\delta_H-\delta_L}{5} > \frac{4-8\delta_H-17\delta_L}{5}$ if $\delta_L < \frac{1-2\delta_H}{8}$, and $x'' = \frac{2-4\delta_H-\delta_L}{5} = \frac{4-8\delta_H-17\delta_L}{5}$ if $\delta_L = \frac{1-2\delta_H}{8}$.

Case 7. $V_{H1} < V_{L1}$ and $V_{H1} \geq \hat{V}_N$, or

$$\delta_L > \delta_H - \frac{1}{7} \text{ and } \delta_L \leq 16\delta_H - 3.$$

There **exists** $x'_L \in [0, x^*_H - \delta_H + \delta_L)$ that improves the payoff of candidate L **if and only if** $\delta_L > \frac{1-2\delta_H}{8}$. The argument here is identical to Case 6.

Case 8. $\hat{V}_N < V_{L0}$ and $V_{H1} < V_{L1}$, or

$$\delta_L > \frac{1-2\delta_H}{4.25} \text{ and } \delta_L > \delta_H - \frac{1}{7}.$$

There **exists** $x'_L \in [0, x^*_H - \delta_H + \delta_L)$ that improves the payoff of candidate L Take $x'_L = x''$. Then, $x_N = O$ and $\hat{V}_L(x'_L) > \bar{V}_L$, so candidate L is better off.

Case 9. $\hat{V}_N < V_{L0}$ and $V_{H1} \geq V_{L1}$, or

$$\delta_L > \frac{1-2\delta_H}{4.25} \text{ and } \delta_L \leq \delta_H - \frac{1}{7}.$$

There **exists** $x'_L \in [0, x^*_H - \delta_H + \delta_L)$ that improves the payoff of candidate L Take $x'_L = x''$. The argument here is identical to Case 8.

Combining Cases 1-9, we obtain conditions (23). □

Proof of Theorem 7. The proof of this theorem is analogous to the proof of Theorem 2. An additional condition (26) has to be satisfied; this condition is more strict than (13).

It remains to be shown that, for small δ_L, (26) is satisfied if and only if δ_H is large enough. Take $\delta_L = 0$ and denote

$$D_2 = F(x^*_H + \delta_H) - 2F(x^*_H - \delta_H). \tag{52}$$

By the Implicit Function Theorem and Assumption $2'$, we have

$$\frac{\partial D_2}{\partial \delta_H} = f_{H+}\left(\frac{\partial x^*_H}{\partial \delta_H} + 1\right) - 2f_{H-}\left(\frac{\partial x^*_H}{\partial \delta_H} - 1\right) = \frac{f_{H-}}{|M|}(4f_{H+}f_L - f_{H+}f_m + 4f_Lf_m) > 0. \tag{53}$$

We have $D_2 < 0$ for δ_H near 0 and $D_2 > 0$ for δ_H near $\frac{1}{2}$. It follows that there exists $\delta_{H0} \in (0, \frac{1}{2})$ such that $D_2 < 0$ if $\delta_H < \delta_H 0$ and $D_2 > 0$ if $\delta_H > \delta_{H0}$. □

Proof of Theorem 8. Consider the following cases.
*C*ase 1. $F(x_H - \delta_H) < 1 - F(x_H + \delta_H)$.

*C*ase 1A. $x_L \in [0, x_H - \delta_H + \delta_L)$. Candidate N has the limit best response $x_N = x_H + \delta_H$. If $x_H - \delta_H + \delta_L > 0$, then candidate L has deviation $x'_L \in (x_L, x_H - \delta_H + \delta_L)$. Let $x_L = 0$ and $x_H = \delta_H - \delta_L$. We have $\tilde{V}_H = F(2\delta_H - \delta_L)$, $\tilde{V}_L = 0$, and $\frac{\partial \tilde{V}_H}{\partial x_H} = f(2\delta_H - \delta_L) - \frac{1}{2}f(0)$. If $\frac{\partial \tilde{V}_H}{\partial x_H} > 0$, candidate H can increase his payoff with some $x'_H > x_H$. If $\frac{\partial \tilde{V}_H}{\partial x_H} \leq 0$, then the peak of f must lie to the left of $2\delta_H - \delta_L$, so f is decreasing on $[2\delta_H - \delta_L, 1]$. Take $x'_L = 1$. Then we have $V_N = F(\frac{x_N + 1 - \delta_L}{2}) - F(\frac{2\delta_H - \delta_L + x_N}{2})$, which is decreasing at $x_N = 1 + \delta_L$. So, candidate N will enter to the left of $1 + \delta_L$, and candidate L will have a positive vote share.

*C*ase 1B. $x_L = x_H - \delta_H + \delta_L$ if $x_H - \delta_H + \delta_L > 0$. Candidate N has the limit best response $x_N = x_H + \delta_H$. Let $x'_L = x_H - \delta_H + \delta_L - e$, with $e > 0$. Then, candidate N will still have the limit best response $x_N = x_H + \delta_H$, and $\tilde{V}'_L = x'_L > F(\frac{x_L}{2}) = \frac{x_L}{2} = \tilde{V}_L$ if e is small enough.

*C*ase 1C. $x_L \in (x_H - \delta_H + \delta_L, x_H + \delta_H - \delta_L)$. Same as Case 1A. Candidate N has the limit best response $x_N = x_H + \delta_H$. We have $\tilde{V}_H = F(x_H + \delta_H)$, which is increasing in x_H.

*C*ase 1D. $x_L \in (x_H + \delta_H - \delta_L, 1)$. Let $V_{N1} = F(x_H - \delta_H) = \lim_{x_N \to_- x_H - \delta_H} V_N$, $V_{N2} = 1 - F(\frac{x_H + x_L + \delta_H + \delta_L}{2}) = \lim_{x_N \to_+ x_L + \delta_L} V_N$, and $V_{N3} = \sup_{x_N \in (x_H + \delta_H, x_L + \delta_L)} V_N$.

Consider the following subcases.

Case 1D1. $V_{N2} > \frac{1}{2}$. Then candidate N will obtain the sole first place. Consider the following subcases.

Case 1D1a. $V_{N2} > V_{N3}$. Then candidate N has limit best response at $x_N = x_L + \delta_L$. If we take $x'_H = x_H + e$ and $e > 0$ is sufficiently small, we will have $V'_{N2} > \max\{\frac{1}{2}, V'_{N3}\}$, with $\tilde{V}'_H > \tilde{V}_H$ and candidate H ranking second, as previously.

Case 1D1b. $V_{N2} = V_{N3}$. Suppose that N has best response $x_N = b(x_H, x_L) \in (x_H + \delta_H, x_L + \delta_L)$. In that case, we must have $f(x_1) = f(x_2)$, where $x_1 = \frac{x_H + x_N + \delta_H}{2}$ and $x_2 = \frac{x_L + x_N - \delta_L}{2}$. Moreover, by Assumption 3, we should have $f'(x_1) > 0$ and $f'(x_2) < 0$. By the Implicit Function Theorem, we should have $\frac{\partial x_1}{\partial x_H} > 0$. As a result, if we take $x'_H = x_H + e$, we should have $\tilde{V}'_H \geq F(\frac{x_H + b(x'_H, x_L) + \delta_H}{2}) > F(\frac{x_H + b(x_H, x_L) + \delta_H}{2}) = \tilde{V}_H$ if $e > 0$ is sufficiently small.

Now let the set $b_0(x_H, x_L)$ contain two limit points $x_N = x_H + \delta_H$ and $x_N = x_L + \delta_L$. If $J = H$, then limit point $x_N = x_H + \delta_H$ is chosen by N, and the vote share $\tilde{V}_H$ is increasing in x_H. If $J = L$, then limit point $x_N = x_L + \delta_L$ is chosen by N. Take $x'_L = x_L + e$. If e is sufficiently small, we should have $V'_{N3} > V_{N3} = V_{N2} > V'_{N2}$, so candidate N will have limit best response $x'_N = x_H + \delta_H$ or best response $b(x_H, x'_L) \in (x_H + \delta_H, x'_L + \delta_L)$. That should increase the vote share of L from zero to some positive value.

Case 1D1c. $V_{N2} < V_{N3}$. Take $x'_H = x_H + e$. Candidate N has either best response $b(x_H, x_L) \in (x_H + \delta_H, x_L + \delta_L)$ or limit best response $x_N = x_H + \delta_H$. In either case, $x'_H = x_H + e$ will increase $\tilde{V}_H$ if e is small (by the argument above).

Case 1D2. $V_{N2} = \frac{1}{2}$. Candidate N will have limit best response at $x_N = x_L + \delta_L$ if $V_H \geq V_{N3}$ (with V_H corresponding to $V_N = V_{N3}$) and a best response $b(x_H, x_L) \in (x_H + \delta_H, x_L + \delta_L)$ or a limit best response $x_N = x_H + \delta_H$ if $V_H < V_{N3}$ (in the latter case, candidate N will rank first). In any case, by the above argument candidate H can increase his rank/vote share with $x'_H = x_L + \delta_L + e$, if e is small enough.

Case 1D3. $V_{N2} < \frac{1}{2}$. We first show that candidate N cannot obtain a sole first place in any equilibrium. Suppose that it is not the case, and candidate N has a best response $b(x_H, x_L) \in (x_H + \delta_H, x_L + \delta_L)$ such that it gives him the sole first place. As $V_{N2} < \frac{1}{2}$, take $x'_L = x_L - e$. We have $\frac{\partial x_2}{\partial x_L} > 0$ by argument similar Case 1D1b. So, $\tilde{V}_L$ will increase and the rank ordering will be preserved if $e > 0$ is sufficiently small. If candidate N has a limit best response $x_N = x_H + \delta_H$ or $x_N = x_H - \delta_H$ such that gives him the sole first place, candidate L can also take $x'_L = x_L - e$ and increase his vote share. As a consequence, candidate N cannot rank first in any equilibrium.

Next we show that candidate N cannot share first place with candidate H in any equilibrium. Consider the contrary. As $V_{N2} < \frac{1}{2}$, candidate N cannot have limit best response at $x_N = x_L + \delta_L$. So, one of the following must be the case. First, candidate N can have limit best response at $x_N = x_H - \delta_H$. Take $x'_L = x_L - e$. Then N will rank first with $x_N = x_H - \delta_H$, and candidate L will increase his vote share. Second, candidate N can have limit best response at $x_N = x_H + \delta_H$ or best response at $x_N \in (x_H + \delta_H, x_L + \delta_L)$. We must have $V_{N1} < \tilde{V}_H = V_{N3} = \tilde{V}_N$. Take $x'_H = x_H + e$. If e is small, then candidate N will not have a limit best response at $x'_N \neq x'_H - \delta_H$ because if $V_{N1'} < V_{N3'}$, so $\tilde{V}'_H > \tilde{V}_H$.

We have established that, in any equilibrium, we must have $\tilde{V}_H > \tilde{V}_N > \tilde{V}_H$. We are now ready to consider different cases of V_{N1}, V_{N2}, and V_{N3}.

Case 1D3a. $V_{N1} > \max\{V_{N2}, V_{N3}\}$. Then, N has limit best response at $x_N = x_H - \delta_H$. Take $x'_H = x_H - e$. If $e > 0$ is small enough, N will continue to have limit best response at $x'_N = x'_H - \delta_H$, with $\tilde{V}'_H > \tilde{V}_H$.

Case 1D3b. $V_{N2} > \max\{V_{N1}, V_{N3}\}$. Then, N has limit best response at $x_L = x_L + \delta_L$. Take $x'_H = x_H + e$. If $e > 0$ is small enough, N will continue to have the same limit best response, with $\tilde{V}'_H > \tilde{V}_H$.

Case 1D3c. $V_{N3} > \max\{V_{N1}, V_{N2}\}$. Then, N has limit best response at $x_N = x_H + \delta_H$ or a best response $b(x_H, x_L) \in (x_H + \delta_H, x_L + \delta_L)$. As $\frac{\partial x_2}{\partial x_L} > 0$, candidate L will increase his share of vote with $x'_L = x_L - e$ if $e > 0$ is small enough.

Case 1D3d. $V_{N1} = V_{N2} > V_{N3}$. If $J = H$[13], then N has limit best response at $x_N = x_H - \delta_H$, and candidate H can deviate $x'_H = x_H - e$. If $e > 0$ is small enough, candidate N will have limit best response $x'_N = x_L + \delta_L$, increasing H's vote share. If $J = L$, then N has limit best response at $x_N = x_L + \delta_L$, and candidate L can deviate $x'_L = x_L + e$. If $e > 0$ is small enough, candidate N will have limit best response $x'_N = x_H - \delta_H$, increasing L's vote share.

Case 1D3e. $V_{N1} = V_{N2} > V_{N3}$. If $J = H$, then N has limit best response at $x_N = x_H - \delta_H$. If $J = L$, then N will have a best response $b(x_H, x_L) \in (x_H + \delta_H, x_L + \delta_L)$ or a limit best response $x_N = x_H + \delta_H$. In any case, candidate N can increase his share of vote with $x'_L = x_L - e$ if $e > 0$ is small enough.

Case 1D3f. $V_{N2} = V_{N3} > V_{N1}$. If $J = H$, then N will have a best response $b(x_H, x_L) \in (x_H + \delta_H, x_L + \delta_L)$ or a limit best response $x_N = x_H + \delta_H$. Then, the vote share of candidate H will increase if $x'_H = x_H + e$ if $e > 0$ is small enough. If $J = L$, then N will have limit best response $x_N = x_L + \delta_L$. If $x'_L = x_L + e$, candidate N will have limit best response $x'_N = x_H + \delta_H$ or a best response $b(x_H, x'_L) \in (x_H + \delta_H, x_L + \delta_L)$, increasing L's share of vote from zero.

Case 1D3g. $V_{N2} = V_{N3} = V_{N1}$. Suppose that candidate N has best response $x_N = b(x_H, x_L) \in (x_H + \delta_H, x_L + \delta_L)$. Then we have $f(\frac{x_H + \delta_H + x_N}{2}) = f(\frac{x_N + x_L - \delta_L}{2})$. We have $V_{N2} = 1 - F(\frac{x_H + x_L + \delta_H + \delta_L}{2})$. As $\frac{x_N + x_L - \delta_L}{2} - \frac{x_H + x_L + \delta_H + \delta_L}{2} > |\delta_L|$ and the distribution f is single-peaked, we have

$$\tilde{V}_L < V_2 - |\delta_L| f\left(\frac{x_N + x_L - \delta_L}{2}\right) < V_1 - |\delta_L| f(x_H - \delta_H) < \tilde{V}'_L$$

with $x'_L = x_H - \delta_H - e$, if $e > 0$ is small enough.[14] Now suppose that N does not have a best response in the interval $(x_H + \delta_H, x_L + \delta_L)$. If $J = L$, then N will have a limit best response $x_N = x_L + \delta_L$. If $x'_L = x_L + e$, with $e > 0$, candidate N will have limit best response $x'_N = x_H + \delta_H$ or a best response $b(x_H, x'_L) \in (x_H + \delta_H, x_L + \delta_L)$, increasing L's share of vote from zero. If $J = H$, we cannot easily determine a payoff-improving deviation for either candidate H or candidate L.

Case 1E. $x_L = 1$. In this case, only V_{N1} and V_{N3} are well-defined. Consider the following cases:

Case 1E1. $V_{N1} > V_{N3}$. In that case, candidate N has limit best response at $x_N = x_H - \delta_H$, and candidate H can increase his vote share with $x_H = x_H - e$, if $e > 0$ is small enough.

Case 1E2. $V_{N1} = V_{N3}$. If $J = H$, then candidate N has limit best response at $x_N = x_H - \delta_H$, and candidate L can increase his vote share with $x'_L = 1 - e$, if $e > 0$ is small enough. Such a deviation will also increase L's vote share if $J = L$ and $\lim_{x_N \to 1 - \delta_L} V_N < V_{N3}$. If the latter is satisfied as equality,

[13] Here and below, we consider $J = H$ as a prelude to the proof of Theorem 9.

[14] It is straightforward to show that $x_H - \delta_H + \delta_L > 0$. Indeed, we have

$$V_1 = V_2 = \int_{\frac{x_H + x_L + \delta_H + \delta_L}{2}}^{\frac{x_N + x_L - \delta_L}{2}} f(x)dx + \int_{\frac{x_N + x_L - \delta_L}{2}}^{1} f(x)dx,$$

where x_N is the (limit) best response chosen when candidate N's vote share is maximized on $(x_H + \delta_H, x_L + \delta_L)$. The last summand is positive, and the first one is greater that $\int_{\max\{0, x_H - \delta_H + \delta_L\}}^{x_H - \delta_H} f(x)dx$, so we must have $\int_0^{\max\{0, x_H - \delta_H + \delta_L\}} f(x)dx > 0$, so $x_H - \delta_H + \delta_L > 0$.

then candidate N should have limit best response at $x_N = 1 + \delta_L$, with $\tilde{V}_L = 0$, and also we should have $f(x_2) \leq f(1)$. As f is assumed to be single-peaked, it must be maximized within the interval $[x_2, 1]$. Thus we should have $1 - \frac{x_H+1+\delta_H+\delta_L}{2} > x_H - \delta_H$, or $x_H - \delta_H + \delta_L > 1 - 2x_H > 0$. It follows that candidate L can choose position $x'_L = x_H - \delta_H + \delta_L - e$ such that $\tilde{V}'_L < 0$ if $e > 0$ is small enough.

*C*ase 1E3. $V_{N1} < V_{N3}$. In that case, candidate N has limit best response at $x_N = x_H + \delta_H$, a limit best response $x_N = 1 + \delta_L$, or a best response $b(x_H, x_L) \in (x_H + \delta_H, x_L + \delta_L)$. In all cases, candidate H can increase his share of vote with $x'_H = x_H + e$.

*C*ase 2. $F(x_H - \delta_H) = 1 - F(x_H + \delta_H)$.

*C*ase 2A. $x_L \in [0, x_H - \delta_H + \delta_L)$. Candidate N has the limit best response $x_N = x_H + \delta_H$. We have $\tilde{V}_L = F(\frac{x_H-\delta_H+x_L+\delta_L}{2})$, which is increasing in x_L; then moving (slightly) to the right, L increases his vote share and does not decrease the rank.

*C*ase 2B. $x_L \in [x_H - \delta_H + \delta_L, x_H + \delta_H - \delta_L]$. Candidate N has two limit best responses: $x_N = x_H + \delta_H$ and $x_N = x_H - \delta_H$. We have $\tilde{V}_L = 0$ if $x_L \in (x_H - \delta_H + \delta_L, x_H + \delta_H - \delta_L)$. If $x_L = x_H - \delta_H + \delta_L$, we have (assuming that $x_H > \delta_H - \delta_L$) $\tilde{V}_L = \frac{1}{2}F(x_H - \delta_H + \delta_L)$. If we take $x'_L = x_L - e$, we must get $\tilde{V}'_L = F(x_H - \delta_H + \delta_L - e)$, which will be greater than $\tilde{V}_L$ if $e > 0$ is small enough.

*C*ase 2C. $x_L \in (x_H + \delta_H - \delta_L, 1]$. This is identical to Case 2A.

□

Proof of Theorem 9. From Theorem 8 it follows that equilibrium in case $J = H$ is only possible if $V_{N1} = V_{N2} = V_{N3}$. If $V_{N1} = V_{N2}$ and the distribution of voter ideal points is symmetric, then we should have $x_H - \delta_H = 1 - \frac{x_H+x_L+\delta_H+\delta_L}{2}$, or $x_L = 2 - 3x_H + \delta_H - \delta_L$.

Let $x_H \in [\frac{1+\delta_H-\delta_L}{3}, \frac{1}{2}]$ and $x_L = 2 - 3x_H + \delta_H - \delta_L$. Then both $V_{N1} = V_{N2}$ increase with x_H, while $V_{N3} = F(\frac{x_H+x_L+\delta_H-\delta_L}{2}) - F(x_H + \delta_H)$ decreases with x_H. At the same time, for $x_H = \frac{1+\delta_H-\delta_L}{3}$ we have $x_H - \delta_H = \frac{x_H+x_L+\delta_H-\delta_L}{2} - x_H - \delta_H$, so $V_{N1} < V_{N3}$, as the density is single-peaked. Finally, for $x = \frac{1}{2}$ we have $V_{N3} = F(\frac{1}{2} + \delta_H - \delta_L) - F(\frac{1}{2} + \delta_H) < F(\frac{1}{2} - \delta_H) = V_{N1}$. It follows that there exists a unique x_H such that $V_{N1} = V_{N2} = V_{N3}$ with $x_L > x_H$.

In that case, candidate N will have limit best response $x_N = x_H - \delta_H$. It is then straightforward to show that candidate H has no payoff-improving deviations. We have $\tilde{V}_H = F(\frac{x_H+\delta_H+x_L-\delta_L}{2}) - F(x_H - \delta_H)$. As f is symmetric and $x_H - \delta_H = 1 - \frac{x_H+\delta_H+x_L+\delta_L}{2}$, $\tilde{V}'_H$ decreases with x'_H, because for $x'_H > x_H$ candidate N will have limit best response $x'_N = x'_H - \delta_H$. Any $x'_H < x_H$ will result in candidate N entering to the right of H, reducing his share of vote. Similarly for L, we have $x_H - \delta_H + \delta_L = 1 - \frac{x_H+\delta_H+x_L-\delta_L}{2}$, so $\tilde{V}'_L < \tilde{V}_L$ for any $x'_L \in [0, x_H - \delta_H + \delta_L)$. Any deviation $x'_L \in (x_H + \delta_H - \delta_L, x_L)$ will result in candidate N entering with limit best response $x'_N = x_L + \delta_L$ and $\tilde{V}'_L = 0$.

□

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