NATIONAL RESEARCH UNIVERSITY

ADVANCED ECONOMETRICS

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Univariate Time Series Models

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Univariate time series modeling

$$Y_1, \ldots, Y_T$$

In the univariate case a series is modeled only in terms of its own past values and some disturbances.

$$Y_t = f(Y_{t-1}, Y_{t-2}, \ldots, \varepsilon_t)$$

$$Lag \quad operator:$$

$$L(Y_t) = Y_{t-1}$$

$$L^2(Y_t) = Y_{t-2}$$

$$L^s(Y_t) = Y_{t-s}$$

$$(1-L)Y_t = Y_t - Y_{t-1} = \Delta Y_t.$$

$$\varepsilon_t \quad is \quad a \quad white \quad noise \quad process$$

$$(E(\varepsilon_t) = 0, \quad \text{var}(\varepsilon_t) = \sigma_\varepsilon^2, \quad \text{cov}(\varepsilon_i, \varepsilon_j) = 0 \quad \forall i \neq j).$$

Stationarity

Def. A stochastic process is said to be strictly stationary if its properties are unaffected by a change of time origin, in other words, the joint probability distribution at any set of times is not affected by an arbitrary shift along the time axis.

Def. A process $\{Y_t\}$ is defined to be weakly stationary if for all t it holds that

$$E\{Y_t\} = \mu < \infty,$$

$$\operatorname{var}\{Y_t\} = E\{(Y_t - \mu)^2\} = \gamma_0 < \infty,$$

$$\operatorname{cov}\{Y_t, Y_{t-k}\} = E\{(Y_t - \mu)(Y_{t-k} - \mu)\} = \gamma_k,\ k = 1,2,3\ldots$$

$$\gamma_k = \operatorname{cov}\{Y_t, Y_{t-k}\} \quad \text{is} \quad \text{k-th} \quad \text{order} \quad \text{autocovariance.}$$

Stationarity. Example

$$Ex.\quad Y_t = Y_{t-1} + \varepsilon_t - random \quad walk,$$

$$Y_t = \varepsilon_t + \varepsilon_{t-1} + Y_{t-2} = \varepsilon_t + \varepsilon_{t-1} + \ldots,$$

$$\mathrm{var}(Y_t) = \sigma_\varepsilon^2 t \Rightarrow process \quad is \quad non \quad stationary,$$

$$\Delta Y_t = \delta + \varepsilon_t - random \quad walk \quad with \quad drift.$$

Stationarity

Def. A stochastic process is said to be strictly stationary if its properties are unaffected by a change of time origin, in other words, the joint probability distribution at any set of times is not affected by an arbitrary shift along the time axis.

COV(Y_t,Y_{t-k}) does not depend upon t.

Def. A process {Y_t} is defined to be weakly stationary if for all t it holds that

$$\gamma_k = \operatorname{cov}\{Y_t, Y_{t-k}\} \quad \text{is} \quad \text{k-th} \quad \text{order} \quad \text{autocovariance.}$$

Def. Autocorrelation function (ACF):

$$\rho_{\mathrm{k}} = \frac{\operatorname{cov}\{Y_t, Y_{t-k}\}}{\operatorname{var}\{Y_t\}} = \frac{\gamma_k}{\gamma_0}$$

Autocorrelation function. Examples

$Ex.1.\quad AR(1)\quad process:\quad Y_t = \delta + \theta Y_{t-1} + \varepsilon_t,$

$\triangleright\ \rho_k = \theta^k \triangleleft.$

$Ex.2.\ AR(2)\quad process:\quad Y_t = \delta + \theta_1 Y_{t-1} + \theta_2 Y_{t-2} + \varepsilon_t.$

$\triangleright$ *By assuming stationarity, the unconditional mean is*

$\mu = \delta/(1-\theta_1-\theta_2).$

$y_t = Y_t - \mu,$

$y_t = \theta_1 y_{t-1} + \theta_2 y_{t-2} + \varepsilon_t$

$y_t y_t = \theta_1 y_t y_{t-1} + \theta_2 y_t y_{t-2} + y_t \varepsilon_t,$

$E(y_t y_t) = \theta_1 E(y_t y_{t-1}) + \theta_2 E(y_t y_{t-2}) + E(y_t \varepsilon_t),$

$\gamma_0 = \theta_1 \gamma_1 + \theta_2 \gamma_2 + \sigma_\varepsilon^2$

Autocorrelation function. Examples

$$y_t = \theta_1 y_{t-1} + \theta_2 y_{t-2} + \varepsilon_t$$

$$y_{t-1} y_t = \theta_1 y_{t-1} y_{t-1} + \theta_2 y_{t-1} y_{t-2} + y_{t-1}\varepsilon_t,$$

$$E(y_{t-1} y_t) = \theta_1 E(y_{t-1} y_{t-1}) + \theta_2 E(y_{t-1} y_{t-2}) + E(y_{t-1}\varepsilon_t),$$

$$\gamma_1 = \theta_1 \gamma_0 + \theta_2 \gamma_1,$$

$$y_{t-2} y_t = \theta_1 y_{t-2} y_{t-1} + \theta_2 y_{t-2} y_{t-2} + y_{t-2}\varepsilon_t,$$

$$E(\mathrm{y}_{t-2} y_t) = \theta_1 E(y_{t-2} y_{t-1}) + \theta_2 E(y_{t-2} y_{t-2}) + E(y_{t-2}\varepsilon_t),$$

$$\gamma_2 = \theta_1 \gamma_1 + \theta_2 \gamma_0$$

Autocorrelation function. Examples

$$\begin{cases} \gamma_0 = \theta_1\gamma_1 + \theta_2\gamma_2 + \sigma_\varepsilon^2, \\ \gamma_1 = \theta_1\gamma_0 + \theta_2\gamma_1, \\ \gamma_2 = \theta_1\gamma_1 + \theta_2\gamma_0, \end{cases}$$

$$\gamma_0 = \frac{(1-\gamma_2)\sigma_\varepsilon^2,}{(1+\gamma_2)(1-\gamma_1-\gamma_2)(1+\gamma_1-\gamma_2)}$$

$$Stationarity \quad conditions \quad for \quad AR(2):$$

$$\gamma_1 + \gamma_2 < 1,$$

$$\gamma_2 - \gamma_1 < 1,$$

$$|\gamma_2| < 1$$

Autocorrelation function. Examples

$$\begin{cases} \gamma_1 = \theta_1\gamma_0 + \theta_2\gamma_1, \\ \gamma_2 = \theta_1\gamma_1 + \theta_2\gamma_0 \end{cases} \Rightarrow$$

$$\begin{cases} \rho_1 = \theta_1 + \theta_2\rho_1, \\ \rho_2 = \theta_1\rho_1 + \theta_2 \end{cases} - Yule - Walker \quad equations \ for \quad AR(2),$$

$$\rho_1 = \frac{\theta_1}{1-\theta_2}, \quad \rho_2 = \frac{\theta_1^2}{1-\theta_2} + \theta_2,$$

$$\rho_k = \theta_1\rho_{k-1} + \theta_2\rho_{k-2}, \quad k = 3,4,\ldots$$

This is a second order difference equation.

The ACF will be damped exponential. $\triangleleft$

Autocorrelation function. Examples

$$Ex.3. \quad MA(1) \quad process: Y_t = \mu + \varepsilon_t + \alpha\varepsilon_{t-1}.$$

$$\triangleright\ \gamma_0 = (1+\alpha^2)\sigma_\varepsilon^2,$$

$$\gamma_1 = \alpha\sigma_\varepsilon^2,$$

$$\gamma_2 = \gamma_3 = \ldots = 0$$

$$\Rightarrow \rho_1 = \frac{\alpha}{1+\alpha^2},$$

$$\rho_2 = \rho_3 = \ldots = 0. \triangleleft$$

General ARMA processes

$$y_t = Y_t - \mu,$$

$$AR(p) \quad process:$$

$$\mathrm{y}_t = \theta_1 y_{t-1} + \theta_2 y_{t-2} + \ldots + \theta_p y_{t-p} + \varepsilon_t,$$

$$MA(q) \quad \text{process}:$$

$$\mathrm{y}_t = \varepsilon_t + \alpha_1 \varepsilon_{t-1} + \ldots + \alpha_q \varepsilon_{t-q},$$

$$ARMA(p,q) \quad \text{mod}\,el:$$

$$\mathrm{y}_t = \theta_1 y_{t-1} + \theta_2 y_{t-2} + \ldots + \theta_p y_{t-p} + \varepsilon_t + \alpha_1 \varepsilon_{t-1} + \ldots + \alpha_q \varepsilon_{t-q}$$

Lag polynomials

$$Lag \quad operator:$$

$$L(Y_t) = Y_{t-1}$$

$$L^s(Y_t) = Y_{t-s}$$

$$AR(1): y_t = \theta y_{t-1} + \varepsilon_t \Leftrightarrow (1-\theta L) y_t = \varepsilon_t$$

$$AR(p): \mathrm{y_t} = \theta_1 y_{t-1} + \theta_2 y_{t-2} + \ldots + \theta_p y_{t-p} + \varepsilon_t \Leftrightarrow$$

$$\theta(L)\mathrm{y_t} = \varepsilon_t,$$

$$\theta(L) = 1 - \theta_1 L - \theta_2 L^2 - \ldots - \theta_p L^p$$