NATIONAL RESEARCH UNIVERSITY

ADVANCED ECONOMETRICS

Lecture 17-24.02.16

Univariate Time Series Models

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General ARMA processes

$$y_t = Y_t - \mu,$$

$$AR(p) \quad process:$$

$$\mathrm{y_t} = \theta_1 y_{t-1} + \theta_2 y_{t-2} + \ldots + \theta_p y_{t-p} + \varepsilon_t,$$

$$MA(q) \quad \text{process}:$$

$$\mathrm{y_t} = \varepsilon_t + \alpha_1 \varepsilon_{t-1} + \ldots + \alpha_q \varepsilon_{t-q},$$

$$ARMA(p,q) \quad \text{mod}\,el:$$

$$\mathrm{y_t} = \theta_1 y_{t-1} + \theta_2 y_{t-2} + \ldots + \theta_p y_{t-p} + \varepsilon_t + \alpha_1 \varepsilon_{t-1} + \ldots + \alpha_q \varepsilon_{t-q}$$

Lag polynomials

$$Lag \quad operator:$$

$$L(Y_t) = Y_{t-1}$$

$$L^s(Y_t) = Y_{t-s}$$

$$AR(1): y_t = \theta y_{t-1} + \varepsilon_t \Leftrightarrow (1 - \theta L) y_t = \varepsilon_t$$

$$AR(p): \mathrm{y_t} = \theta_1 y_{t-1} + \theta_2 y_{t-2} + \ldots + \theta_p y_{t-p} + \varepsilon_t \Leftrightarrow$$

$$\theta(L)\mathrm{y_t} = \varepsilon_t,$$

$$\theta(L) = 1 - \theta_1 L - \theta_2 L^2 - \ldots - \theta_p L^p$$

Invertability of Lag polynomials

$$AR(1): y_t = \theta y_{t-1} + \varepsilon_t \Leftrightarrow (1-\theta L) y_t = \varepsilon_t,$$

$$(1-\theta L)^{-1}(1-\theta L) y_t = (1-\theta L)^{-1}\varepsilon_t,$$

$$(1-\theta L)^{-1} = \sum_{j=0}^{\infty} \theta^j L^j \quad if \quad |\theta| < 1,$$

$$(|\theta| < 1 \quad is \quad the \quad condition \quad for \quad the \quad invertibility),$$

$$y_t = (1-\theta L)^{-1}\varepsilon_t,$$

$$y_t = \sum_{j=0}^{\infty} \theta^j \varepsilon_{t-j}.$$

The MA representation is convenient to determine variance and covariance.

Invertability of Lag polynomials

$$AR(2): y_t = \theta_1 y_{t-1} + \theta_2 y_{t-2} + \varepsilon_t \Leftrightarrow$$

$$(1 - \theta_1 L - \theta_2 L^2) y_t = \varepsilon_t,$$

$$(1 - \varphi_1 L)(1 - \varphi_2 L) y_t = \varepsilon_t,$$

The condition for invertability of $(1 - \varphi_1 L)(1 - \varphi_2 L)$:

both $(1 - \varphi_1 L)$ *and* $(1 - \varphi_2 L)$ *must be invertible.*

Thus, the requirement for invertibility:

$$|\varphi_1| < 1, \quad |\varphi_2| < 1.$$

Invertability of Lag polynomials

$$AR(2): y_t = \theta_1 y_{t-1} + \theta_2 y_{t-2} + \varepsilon_t \Leftrightarrow$$

$$(1-\varphi_1 L)(1-\varphi_2 L) y_t = \varepsilon_t,$$

The characteristic equation:

$$(1-\varphi_1 z)(1-\varphi_2 z) = 0.$$

Conditions for invertability

$$|\varphi_1| < 1, \quad |\varphi_2| < 1 \quad \textit{correspond to}$$

$$|z_1| > 1, \quad |z_2| > 1.$$

The AR polinomial is invertable if and only if the process is stationary.

Invertability of MA polinomial

$$MA(1): y_t = \varepsilon_t + \alpha\varepsilon_{t-1} \Leftrightarrow$$

$$y_t = (1+\alpha L)\varepsilon_t,$$

$$(1+\alpha L)^{-1} = \sum_{j=0}^{\infty}(-\alpha)^j L^j,$$

$$y_t = \alpha\sum_{j=0}^{\infty}(-\alpha)^j y_{t-j-1} + \varepsilon_t$$

$$MA(q): y_t = \alpha(L)\varepsilon_t.$$

A necessary condition for the $AR(\infty)$ *representation: invertability of* $\alpha(L)$.

Stationarity and unit root

$ARMA(p,q)$ model:

$\theta(L)y_t = \alpha(L)\varepsilon_t,$

corresponds to a stationary process if and only if the solutions of equations

$\theta(z) = 0$

satisfy conditions: $|z_j| > 1 \quad \forall \quad j = 1,\ldots,p.$

Special case: $\theta(L) = \theta^*(L)(1-L),$

$\theta^*(L)$ is an invertable polynomial.

$\theta^*(L)(1-L)y_t = \alpha(L)\varepsilon_t \Rightarrow$

$\theta^*(L)\Delta y_t = \alpha(L)\varepsilon_t$ is a stationary process.

Stationarity and unit root

A stationary series is denoted by $I(0)$.

A series which become stationary after first differencing is said to be integrated of order one, denoted $I(1)$.

A series which become stationary only after second differencing is said to be integrated of order two, denoted $I(2)$, *etc.*

Testing for Unit Roots

$$AR(1): Y_t = \delta + \theta Y_{t-1} + \varepsilon_t$$

$$H_0 : \theta = 1 \quad (a \quad unit \quad root)$$

$$\mathrm{H}_1 : |\theta| < 1 \quad (stationarity)$$

$$Dickey - Fuller \quad test:$$

$$DF = \frac{\hat{\theta} - 1}{se(\hat{\theta})}$$

Testing for Unit Roots

$$More\ convenient\quad procedure$$

$$\Delta Y_t = \delta + (\theta - 1)Y_{t-1} + \varepsilon_t$$

$$\theta = 1 \quad corresponds$$

$$\Delta Y_t = \delta + \varepsilon_t - \quad random \quad walk \quad with \quad drift.$$

$$Y_t - difference \quad stationary.$$

Testing for Unit Roots

It is possible that nonstationary is caused by the presence of the deterministic time trend :

$$Y_t = \delta + \theta Y_{t-1} + \gamma t + \varepsilon_t$$

$$with \quad |\theta| < 1 \quad and \quad \gamma \neq 0.$$

$$Y_t - trend \quad stationary.$$

Usual DF test:

$$\Delta Y_t = \delta + (\theta - 1) Y_{t-1} + \gamma t + \varepsilon_t,$$

different critical values of DF statistics: $\tau_0, \tau_\mu, \tau_\tau$.

KPSS test

Kwiatkowski, D., Phillips, P. C., Schmidt, P., & Shin, Y. (1992). Testing the null hypothesis of stationarity against the alternative of a unit root: How sure are we that economic time series have a unit root?. *Journal of econometrics*, *54*(1), 159-178.

$$y_t = \beta t + r_t + \varepsilon_t$$

$$r_t = r_{t-1} + u_t$$

$$H_0 : \sigma_u^2 = 0 \, (stationarity)$$

$$H_1 : \sigma_u^2 \neq 0 \, (non\ stationarity),$$

KPSS test

$$H_0 : \sigma_u^2 = 0 \ (stationarity)$$

$$H_1 : \sigma_u^2 \neq 0 \ (non\ stationarity),$$

$$OLS, \quad residuals \quad e_t, \quad S_t = \sum_{s=1}^{T} e_s,$$

$$KPSS = \sum_{s=1}^{T} S_t^2 / \hat{\sigma}^2.$$

$$The \quad asymptotic \quad distribition \quad is \quad non-standard,$$

$$and \quad Kwiatkowski \quad et \quad al. \quad report \quad KPSS_{0.05}^{cr} = 0.146.$$

$$The \quad same \ algorithm \quad for \quad model \quad without \quad trend,$$

$$KPSS_{0.05}^{cr} = 0.463.$$

Testing for Unit Roots in Higher Order AR Models

$$AR(2) \quad process: \quad Y_t = \delta + \theta_1 Y_{t-1} + \theta_2 Y_{t-2} + \varepsilon_t,$$

$$\Delta Y_t = \delta + (\theta_1 + \theta_2 - 1)Y_{t-1} - \theta_2 \Delta Y_{t-1} + \varepsilon_t.$$

$$H_0 : \pi \equiv \theta_1 + \theta_2 - 1 = 0 \quad (unit \quad root)$$

$$AR(p) \quad process: \quad Y_t = \delta + \theta_1 Y_{t-1} + \ldots + \theta_p Y_{t-p} + \varepsilon_t,$$

$$\Delta Y_t = \delta + \pi Y_{t-1} + c_1 \Delta Y_{t-1} + \ldots + c_{p-1} \Delta Y_{t-p+1} + \gamma t + \varepsilon_t.$$

Augmented Dickey–Fuller test

$$H_0 : \pi \equiv \theta_1 + \ldots + \theta_p - 1 = 0 \quad (unit \quad root).$$

Test statistics $\tau_0, \tau_\mu, \tau_\tau$.

Nonparametric Philips and Perron test.

Dolado, Jenkinson, Sosvilla-Rivero testing for Unit Roots

$$Step1. \quad \Delta Y_t = \delta + \gamma t + \pi Y_{t-1} + c_1 \Delta Y_{t-1} + \ldots + c_{p-1} \Delta Y_{t-p+1} + \varepsilon_t .$$

$$H_0 : \pi = 0 \quad (unit \quad root, \quad DS),$$

$$H_1 : \pi < 0 \quad (TS).$$

$$If \quad t < \tau_\tau (H_0 \quad is \quad rejected) \Rightarrow TS, \quad no\, unit \quad root.$$

$$If \quad t > \tau_\tau \quad (H_0 \quad is \quad not \quad rejected) \quad go \quad to \quad step \quad 2.$$

$$Step2. \quad \Delta Y_t = \delta + \gamma t + c_1 \Delta Y_{t-1} + \ldots + c_{p-1} \Delta Y_{t-p+1} + \varepsilon_t .$$

$$H_0 : \quad \gamma = 0,$$

$$H_1 : \gamma \neq 0.$$

$$If \quad H_0 \quad is \quad rejected, \quad stop.$$

$$If \quad H_0 \quad is \quad not \quad rejected, \quad go \quad to \quad step \quad 3.$$

Dolado, Jenkinson, Sosvilla-Rivero testing for Unit Roots

$$Step3. \quad \Delta Y_t = \delta + \pi Y_{t-1} + c_1 \Delta Y_{t-1} + \ldots + c_{p-1} \Delta Y_{t-p+1} + \varepsilon_t .$$

$$H_0 : \pi = 0 \quad (unit \quad root, \quad DS),$$

$$H_1 : \pi < 0 \quad (const).$$

$$If \quad t < \tau_\mu (H_0 \quad is \quad rejected) \Rightarrow no\, unit \quad root, \quad no \quad time \quad trend.$$

$$If \quad t > \tau_\mu \quad (H_0 \quad is \quad not \quad rejected) \quad go \quad to \quad step \quad 4.$$

$$Step4. \quad \Delta Y_t = \delta + c_1 \Delta Y_{t-1} + \ldots + c_{p-1} \Delta Y_{t-p+1} + \varepsilon_t .$$

$$H_0 : \quad \delta = 0,$$

$$H_1 : \delta \neq 0.$$

$$If \quad H_0 \quad is \quad rejected, \quad stop.$$

$$If \quad H_0 \quad is \quad not \quad rejected, \quad go \quad to \quad step \quad 5.$$

Dolado, Jenkinson, Sosvilla-Rivero testing for Unit Roots

$$Step5. \quad \Delta Y_t = \pi Y_{t-1} + c_1 \Delta Y_{t-1} + \ldots + c_{p-1} \Delta Y_{t-p+1} + \varepsilon_t.$$

$$H_0 : \pi = 0 \quad (unit \quad root, \quad DS),$$

$$H_1 : \pi < 0 \quad (stationarity).$$

$$If \quad t < \tau_0 (H_0 \quad is \quad rejected) \Rightarrow no\ unit \quad root.$$

$$If \quad t > \tau_0 \quad (H_0 \quad is \quad not \quad rejected) \Rightarrow unit \quad root..$$

Choosing a model

Autocorrelation function (ACF):

$$\rho_k = \frac{\operatorname{cov}\{Y_t, Y_{t-k}\}}{\operatorname{var}\{Y_t\}} = \frac{\gamma_k}{\gamma_0},$$

Partial Autocorrelation function (PACF).

The k-th order sample partial autocorrelation coefficient is the estimate for θ_k *in an* $AR(k)$ model.

$AR(2)$ *process*: $Y_t = \delta + \theta_1 Y_{t-1} + \theta_2 Y_{t-2} + \varepsilon_t,$

Estimating $Y_t = \delta + \theta_1 Y_{t-1} + \varepsilon_t$ *gives* $\hat{\theta}_{11},$

estimating $Y_t = \delta + \theta_1 Y_{t-1} + \theta_2 Y_{t-2} + \varepsilon_t$ *gives* $\hat{\theta}_{22}.$

Partial Autocorrelation Function

$AR(p)$ *process*: $PACF = 0$ *if* $k > p$.

$MA(q)$ *process* $= AR(\infty)$ *if* MA *polynomial is invertible*,

PACF coefficients will damp toward zero.

Choosing a model

AR(p) process:

1) **An ACF is infinite in extend (it tails off),**
2) **A PACF is close to zero for lags larger than p**

MA(q) process:

1) **An ACF is close to zero for lags larger than q,**
2) **A PACF is infinite in extend (it tails off),**

Diagnostic Checking

ARMA(p, q), ARMA(p + 1, q), ARMA(p, q + 1) or ???

Idea: residuals of an adequate model should be approximately white noise.

The significance of residual autocorrelation is often checked by comparing with $\pm 2/\sqrt{T}$

Diagnostic Checking

$H_0 : ARMA(p,q)$

$Ljung-Box \quad portmanteau \quad test \quad statistics:$

$$Q_K = T(T+2)\sum_{k=1}^{K} \frac{1}{T-k} r_k^2,$$

r_k^2 are the estimated autocorrelation coefficient of the residuals $\hat{\varepsilon}_t$ and K is a number chosen by the researcher.

$$H_0 : ARMA(p,q) \Rightarrow Q_K \sim \chi^2(K-p-q-1).$$

Criteria for model selection

Akaike's Information Criterion (AIC):

$$AIC = \ln\hat{\sigma}^2 + 2\frac{p+q+1}{T}.$$

Schwarz's Bayesian Information Criterion (BIC):

$$BIC = \ln\hat{\sigma}^2 + \frac{p+q+1}{T}\ln T.$$

Estimation of ARIMA Models

ARIMA(p, d, q)

1) **d – order of difference (after UR testing).**
2) **Analysis of ACF, PACF → p,q**
3) **AIC, BIC**
4) **Estimation of parameters**
5) **AR – OLS**
6) **MA – ML**
7) **ARMA - ML**