A simple solution to the Hotelling problem

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Abstract

I show that in the original two-stage Hotelling model with linear transportation cost, the transportation-efficient locations pair (1/4, 3/4) is the only symmetric locations pair that is induced by a self-enforcing agreement between the two firms.

Keywords: Hotelling, self-enforcing agreements, extensive-form rationalizability.

1 Introduction

The solution of the original formulation of the Hotelling problem has been a long-standing issue in economics. Hotelling (1929) predicted that the two firms, in the attempt to acquire a competitive advantage before the pricing stage, would converge to the middle of the spectrum — the so-called principle of minimum differentiation. Many decades later, d'Aspremont et al. (1979) found a mistake in Hotelling's argument. Hotelling did not consider that, if the firms get too close to the middle, they have the incentive to undercut a sufficiently high price of the competitor, so to conquer the whole market. Because

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of this tendency, there is no subgame perfect equilibrium in pure strategies. To obtain one, D'Aspremont et al. make a radical modification to the model: they impose a quadratic instead of linear transportation cost. This makes serving far-away customers increasingly expensive in terms of size of the undercut. With this, they obtain a unique equilibrium in every subgame and a unique subgame perfect equilibrium for the whole game, where the firms stay at the opposite extremes of the spectrum — the so-called principle of maximum differentiation. Economides (1986) has observed that this extreme result is really an artifact of the convexity of the transportation cost: the firms will move a bit closer to the middle when the convexity is reduced. However, Economides' analysis is constrained by the focus on subgame perfect equilibrium in pure strategies, which soon ceases to exist as convexity diminishes. Osborne and Pitchik (1987) go back to linear transportation cost and look for a subgame perfect equilibrium with numerical methods. They argue that a subgame perfect equilibrium exists, where the firms pick locations at which there is no pure pricing equilibrium.[1] Less confidently, they also speculate that the subgame perfect equilibrium is unique.

I propose an alternative solution concept that yields a simple and natural solution with a clear interpretation. I look for (possibly incomplete) self-enforcing agreements between the two firms, in the sense of Catonini (2019b). It turns out that the only symmetric locations pair that is induced by a self-enforcing agreement is $(1/4, 3/4)$, followed by the unique pure pricing equilibrium of the subgame. Note that $(1/4, 3/4)$ is the transportation-efficient locations pair. This solution needs not be strictly interpreted as explicit collusion: it is likely to emerge also between firms that do not engage in pre-play communication, but realize the risk of a price war that relocating towards the middle entails.

The notion of self-enforceability of Catonini (2019b) is based on forward induction reasoning. Given a possibly incomplete agreement among players,

[1] This locations pair is approximately $(0.27, 0.73)$, very close to the $(1/4, 3/4)$ solution proposed here, but qualitatively very different: after $(1/4, 3/4)$ there is a unique, pure pricing equilibrium.

each player tentatively believes that the opponents are rational and believe in the agreement; that the opponents believe that everyone else is rational and believes in the agreement; and so on. But at some history, common belief in rationality and in the agreement may be impossible, because, say, a player has made a move that is not rational under belief in the agreement. In this case, the opponents will keep the belief that our player is rational, and drop the belief that our player believes in the agreement. On the one hand, this relaxes the coordination requirements on off-path beliefs; on the other hand, strategic reasoning restricts them. For these reasons, self-enforcing agreements depart from subgame perfect equilibria.

The outcomes that can be achieved with a self-enforcing agreement are characterized by Self-Enforcing Sets (Catonini 2019b), which will be used for the analysis. SES's refine extensive form-rationalizability[2] by assuming (partial) coordination among players. Extensive-form-rationalizability does not restrict by itself the set of possible location pairs. SES's generically refine also Extensive-Form Best Response Sets (Battigalli and Friedenberg 2012). EFBRS's are based on the hypothesis that, in the situation depicted above, the opponents drop the belief that our player is rational.[3] This drastically reduces the refinement power: EFBRS's only rule out locations pairs where the firms are very close to $(1/2, 1/2)$.

The intuition for why $(1/4, 3/4)$ is the unique symmetric solution is the following. At locations pairs (a_1, a_2) with $a_1 \leq 1/4$ and $a_2 \geq 3/4$, there is a unique pure pricing equilibrium, which is also the only rationalizable price pair. Given the location of the competitor, the closer a firm is to the middle, the higher its equilibrium profit. Therefore, if the firms were to agree on a locations pair $(a_1, 1 - a_1)$ with $a_1 < 1/4$, each firm could profitably relocate towards the middle. At locations pairs $(a_1, a_2) \neq (1/4, 3/4)$ with $a_1 \geq 1/4$ and

[2]See Pearce (1984), Battigalli (1997), Battigalli and Siniscalchi (2002) for different formulations of extensive-form rationalizability.

[3]EFRBS's capture the behavioral implications of Strong-Δ-Rationalizability (Battigalli 2003, Battigalli and Siniscalchi 2003) across all first-order belief restrictions. Strong-Δ-Rationalizability introduces belief restrictions in the algorithm of extensive-form rationalizability, but by doing so, it does *not* refine extensive-form rationalizability.

$a_2 \leq 3/4$ there is no pure pricing equilibrium, and the rationalizable prices include undercuts. Therefore, if the firms were to agree on a locations pair $(a_1, 1-a_1)$ with $a_1 \in (1/4, 1/2)$, they would also have to agree on a set of prices closed under rational behavior (Basu and Weibull, 1991) that entails the possibility of undercuts.[4] Then, each firm would rather "give in" and move outwards, to a location where undercutting is no more rationalizable for the competitor. At $(1/4, 3/4)$, firms have no reason to undercut each other on path, while a deviation towards the middle entails this risk. In particular, the largest SES that induces locations $(1/4, 3/4)$ threatens to fix any price that the deviator has no incentive to undercut (a weaker condition than undercutting some rationalizable price of the deviator). An agreement that induces the locations pair $(1/4, 3/4)$ can also be coarser than a SES: for instance, firms can just agree to keep the region $(1/4, 3/4)$ as a buffer zone that no firm should enter, leaving to strategic reasoning the convergence to $(1/4, 3/4)$.

The paper is organized as follows. Section 2 introduces the notation and well-known facts about best replies and equilibria of the pricing stage, and the notion of Self-Enforcing Set. Section 3 shows the existence of a SES where the firms locate at $(1/4, 3/4)$. Section 4 shows the non-existence of SES's where the firms pick any other symmetric locations pair. The well-known discountinuity of the profit function entails that some conjectures over prices have no best reply. However, this is not the fundamental reason for the absence of a subgame perfect equilibrium in pure strategies, nor it affects the picture of SES's. Therefore, some arguments in Sections 3 and 4 are provided for a discretized version of the model where every conjecture has a best reply. The technical complications that arise with a continuum of prices are tackled in the Appendix. The Appendix also contains a partial characterization of the EFBRS's of the model.

[4] Firms do not agree on a price distribution. After all, they would lack the strict incentive to implement it, against the spirit of self-enforceability. In this context, alternate classical justifications of mixed equilibrium seem hardly plausible as well.

2 Preliminaries

2.1 Model

Two firms, $i = 1, 2$, sell the same good at locations a_1 and a_2 of a continuum of buyers of measure 1.[5] There is no cost of production. Every buyer buys one unit from one of the two firms, except when both prices are "prohibitively high" — the precise upper bound is immaterial for the analysis. The payoff of buyer $j \in [0, 1]$ when she buys from firm i is $-p_i - |j - a_i|$, where p_i is the price fixed by firm i. Suppose that a buyer chooses at random one of the two firms when indifferent. The two firms first choose simultaneously a_1 and a_2, and then, after observing (a_1, a_2), fix prices simultaneously.

2.2 Pricing stage

Best replies Fix (a_1, a_2). If $a_1 = a_2$, there is Bertrand competition and the unique equilibrium and rationalizable price pair is $(0, 0)$. Else, suppose without loss of generality that $a_1 < a_2$. Then, firm 1 faces demand

$$D_1(p_1, p_2) = \begin{cases} 0 & \text{if } p_1 - p_2 > a_2 - a_1 \\ \frac{a_1}{2} & \text{if } p_1 - p_2 = a_2 - a_1 \\ \frac{a_1+a_2}{2} + \frac{p_2-p_1}{2} & \text{if } a_1 - a_2 < p_1 - p_2 < a_2 - a_1 \\ a_2 + \frac{1-a_2}{2} & \text{if } p_2 - p_1 = a_2 - a_1 \\ 1 & \text{if } p_2 - p_1 > a_2 - a_1 \end{cases} .$$

Fix p_2. Suppose that p_1 is such that $a_1 - a_2 < p_1 - p_2 < a_2 - a_1$. Then, the profit of firm 1 is

$$\pi_1(p_1, p_2) = p_1 \left(\frac{a_1 + a_2}{2} + \frac{p_2 - p_1}{2} \right). \tag{1}$$

[5] This parametrization of the model corresponds to the one of Osborne and Pitchik (1987), i.e., to the choice of $c = 1$ and $l = 1$ in d'Aspremont et al. (1979). Note: in those two papers the position of firm 2 is identified by the distance $b = 1 - a_2$ from buyer $j = 1$.

The first-order condition yields

$$p_1^F(p_2) = \frac{a_1 + a_2}{2} + \frac{p_2}{2}. \tag{2}$$

For $p_1^F(p_2)$ to be the best reply to p_2, two conditions must be verified. First, that $p_1^F(p_2) - p_2 < a_2 - a_1$, otherwise firm 1 obtains no demand.[6] This yields

$$p_2 > 3a_1 - a_2 =: \underline{p}_2.$$

Second, firm 1 should not make a higher profit by taking the whole market with a price slightly below $p_2 - (a_2 - a_1)$. The supremum of firm 1's profit with $p_1 < p_2 - (a_2 - a_1)$ is $p_2 - (a_2 - a_1)$. Hence, we must have

$$\pi_1(p_1^F(p_2), p_2) \geq p_2 - (a_2 - a_1),$$

which yields

$$p_2 \leq 4 - a_2 - a_1 - 4\sqrt{1 - a_2} =: \overline{p}_2. \tag{3}$$

This also implies $p_1^F(p_2) - p_2 > a_1 - a_2$ (or $p_1^F(p_2) - p_2 = a_1 - a_2$, but then $a_2 = 1$, thus $p_1^F(p_2)$ brings demand 1).[7] So, when $\underline{p}_2 < p_2 \leq \overline{p}_2$, $p_1^F(p_2)$ is the best reply to p_2, and substituting (2) into (1), the optimal profit reads

$$\pi_1(p_1^F(p_2), p_2) = \frac{1}{2}\left(p_1^F(p_2)\right)^2. \tag{4}$$

When $p_2 > \overline{p}_2$, firm 1 prefers to fix p_1 slightly below $p_2 - (a_2 - a_1)$ with respect to $p_1^F(p_2)$ (or to a price slightly below $p_2 + (a_2 - a_1)$). When $p_2 \leq \underline{p}_2 \leq \overline{p}_2$, firm 1 prefers to fix p_1 slightly below $p_2 + (a_2 - a_1)$ with respect to $p_1^F(p_2)$ or to a price slightly below $p_2 - (a_2 - a_1)$. When $p_2 \leq \overline{p}_2 < \underline{p}_2$, there is $\overline{p}_2' < \overline{p}_2$ such that firm 1 prefers to fix p_1 slightly below $p_2 - (a_2 - a_1)$ if $p_2 > \overline{p}_2'$, and slightly below $p_2 + (a_2 - a_1)$ if $p_2 < \overline{p}_2'$ — see Appendix A.1 for details.

The analogous conditions for firm 2 given p_1 can be obtained by substitut-

[6] Or $a_1/2$ in case of equality, however $p_1^F(p_2)$ still does worse than some $p_1 < p_2 + (a_2 - a_1)$.

[7] To see this, rewrite (3) as $p_2 \leq 3a_2 - a_1 + 4(1 - a_2) - 4\sqrt{1 - a_2}$; then, $p_2 < 3a_2 - a_1$ if $a_2 < 1$, which is equivalent to $p_1^F(p_2) - p_2 > a_1 - a_2$.

ing a_1 with $1-a_2$ and a_2 with $1-a_1$. So we have

$$\begin{aligned} p_2^F(p_1) &: = 1 - \frac{a_1+a_2}{2} + \frac{p_1}{2}, \\ \underline{p}_1 &: = 2 - 3a_2 + a_1, \\ \overline{p}_1 &: = 2 + a_1 + a_2 - 4\sqrt{a_1}. \end{aligned}$$

Note that, if $a_1 + a_2 \geq 1$, $\overline{p}_1 \leq \overline{p}_2$ and $\underline{p}_1 \leq \underline{p}_2$.

Pure equilibrium The only candidate pure equilibrium is $(p_1^*, p_2^*) = \left(p_1^F(p_2^*), p_2^F(p_1^*)\right)$; that is

$$(p_1^*, p_2^*) = \left(\frac{2+a_1+a_2}{3}, \frac{4-a_1-a_2}{3}\right).$$

Any other price pair cannot be an equilibrium: when $p_1^F(p_2)$ is not a best reply to p_2, there is no best reply to p_2. Also in a fine discretization of the model, if the best reply to p_2 is slightly below $p_2 - (a_2 - a_1)$ or $p_2 + (a_2 - a_1)$ (instead of being close to $p_1^F(p_2)$), p_2 is not a best reply to it.

So, we have to check at which locations pairs p_1^* best replies to p_2^* and vice versa. Conditions $p_2^* > \underline{p}_2$ and $p_1^* > \underline{p}_1$ yield

$$a_1 < \frac{2}{5} + \frac{1}{5}a_2; \tag{5}$$

$$a_2 > \frac{2}{5} + \frac{1}{5}a_1. \tag{6}$$

Conditions $p_2^* \leq \overline{p}_2$ and $p_1^* \leq \overline{p}_1$ yield

$$a_1 \leq 4 - a_2 - 6\sqrt{1-a_2}; \tag{7}$$

$$a_2 \geq 6\sqrt{a_1} - 2 - a_1. \tag{8}$$

Inequalities (7) and (8) are always satisfied for $a_1 \leq 1/4$ and $a_2 \geq 3/4$. Also, they are satisfied for $a_1 \in (1/4, \widehat{a}_1]$ and $a_2 \geq 6\sqrt{a_1} - 2 - a_1 \geq 3/4$, where $\widehat{a}_1 \simeq 0.30$ is the solution of $1 = 6\sqrt{a_1} - 2 - a_1$, and symmetric pairs. At all these locations, inequalities (5) and (6) are satisfied, so (p_1^*, p_2^*) is an equilibrium.

2.3 Extensive-form rationalizability, Self-Enforcing Sets

A *plan of actions* (or *reduced strategy*) is a function $s_i : \{\emptyset\} \cup [0,1] \to [0,\infty)$, where $s_i^{\emptyset} := s_i(\emptyset) \in [0,1]$ is the prescribed location and $s_i(a_{-i})$ for $a_{-i} \in [0,1]$ is the price prescribed for the subgame with locations $(s_i^{\emptyset}, a_{-i})$. Let S_i denote the set of all plans of actions, which I will call just "plans" for brevity. Let $S_i(a_i)$ denote the set of all $s_i \in S_i$ with $s_i^{\emptyset} = a_i$. For any $\overline{S}_{-i} \subseteq S_{-i}$, let $\Delta(\overline{S}_{-i})$ denote the set of Borel probability measures ν over S_{-i} such that $\nu(B) = 1$ for some Borel subset $B \subseteq \overline{S}_{-i}$. Let $\Delta^H(S_{-i})$ be the set of all arrays of beliefs $\mu_i = (\mu_i(\cdot|h))_{h\in\{\emptyset\}\cup[0,1]^2}$ of firm i over S_{-i} such that for each $a = (a_i, a_{-i}) \in [0,1]^2$, $\mu_i(\cdot|a) \in \Delta(S_{-i}(a_{-i}))$ satisfies the chain rule of probability; that is, for each Borel subset $\overline{S}_{-i}$ of $S_{-i}(a_{-i})$, $\mu_i(\overline{S}_{-i}|a) \cdot \mu_i(S_{-i}(a_{-i})|\emptyset) = \mu_i(\overline{S}_{-i}|\emptyset)$.[8] For any $\overline{S}_{-i} \subset S_{-i}$, let $\Delta^H(\overline{S}_{-i})$ denote the set of all $\mu_i \in \Delta^H(S_{-i})$ that *strongly believe* $\overline{S}_{-i}$, i.e., such that $\mu_i(\overline{S}_{-i}|\emptyset) = 1$ and $\mu_i(\overline{S}_{-i} \cap S_{-i}(a_{-i})|a) = 1$ for each $a = (a_i, a_{-i}) \in [0,1]^2$ with $\overline{S}_{-i} \cap S_{-i}(a_{-i}) \neq \emptyset$.[9] Finally, let $\rho_i(\mu_i)$ be set of all *sequential best replies* to μ_i; that is, the (possibly empty) set of all $s_i \in S_i$ such that (i) s_i maximizes expected profit given $\mu_i(\cdot|\emptyset)$, and (ii) for each $a_{-i} \in [0,1]$, $s_i(a_{-i})$ maximizes expected profit given the distribution over firm $-i$'s prices induced by $\mu_i(\cdot|s_i^{\emptyset}, a_{-i})$.

Now, let $S_i^0 := S_i$, and define S_i^n inductively as

$$S_i^n := \left\{ s_i \in S_i^{n-1} : \exists \mu_i \in \Delta^H(S_{-i}^{n-1}), s_i \in \rho_i(\mu_i) \right\}.$$

Finally, let $S_i^{\infty} := \cap_n S_i^n$ be the set of *extensive-form rationalizable* plans of firm i. In Appendix A I will show that all locations pairs are compatible with extensive-form rationalizability.

[8]In a finite setting, such an array of beliefs is called Conditional Probability System. In an infinite setting, there is no consensus on how to strengthen the chain rule to condition on probability zero events. However, all the arguments in the paper will depend only on arrays of belief μ_i such that $\mu_i(S_{-i}(a_{-i})|\emptyset) = 1$ for some $a_{-i} \in [0,1]$, so no strengthening of the chain rule would have any impact on the analysis.

[9]I am implicitly assuming that $\overline{S}_{-i}$ is a Borel set; if this is not the case, interpret $\mu_i(\overline{S}_{-i}|\emptyset) = 1$ as $\mu_i(B|\emptyset) = 1$ for a Borel subset B of $\overline{S}_{-i}$ (and analogously for $\mu_i(\overline{S}_{-i} \cap S_{-i}(a_{-i})|a) = 1$): this avoids proving formally that the strongly believed sets are Borel.

Now, fix a locations pair $a \in [0,1]^2$. For each $i = 1, 2$, let $P_i^0(a) := [0, \infty)$. For each $n > 0$, let $P_i^n(a)$ be the set of prices that best reply to a Borel probability measure over $P_{-i}^{n-1}(a)$. Finally, let $P_i^\infty(a) := \cap_{n>0} P_i^n(a)$. So, $P_i^n(a)$ are the prices of firm i that survive n steps of rationalizability in the subgame with locations pair a, and $P_i^\infty(a)$ are the rationalizable prices. Observe by induction that for each $a = (a_i, a_{-i})$, if $S_i^n \cap S_i(a_i) \neq \emptyset$ for each $i = 1, 2$, then

$$\{p_i : \exists s_i \in S_i^n \cap S_i(a_i), s_i(a_{-i}) = p_i\} \subseteq P_i^n(a)$$

for each $i = 1, 2$. Then, if $S_i^n \cap S_i(a_i) \neq \emptyset$ for each $i = 1, 2$ and all $n > 0$,

$$\{p_i : \exists s_i \in S_i^\infty \cap S_i(a_i), s_i(a_{-i}) = p_i\} \subseteq P_i^\infty(a)$$

for each $i = 1, 2$. So, the extensive-form rationalizable pairs of plans prescribe only rationalizable price pairs of the subgames they reach.

In games with two players or two stages, a Self-Enforcing Set is a subset of extensive-form rationalizable plans $S^{es} = S_1^{es} \times S_2^{es} \subset S^\infty$ that satisfies two conditions: Realization-Strictness and Self-Justifiability.[10] The two conditions are translated here for the present context. For a pair of plans (s_1, s_2), let $\zeta(s_1, s_2)$ denote the induced path. For each $i = 1, 2$:

Realization-strictness: for each $\mu_i \in \Delta^H(S_{-i}^{es})$, $\zeta(\rho_i(\mu_i) \times S_{-i}^{es}) \subseteq \zeta(S^{es})$;

Self-Justifiability: for each $s_i \in S_i^{es}$, $s_i \in \rho_i(\mu_i)$ for some $\mu_i \in \Delta^H(S_{-i}^{es})$.[11]

Realization-strictness means that players have no incentive to leave the paths induced by the SES when they believe in it. Self-Justifiability means that every plan in the SES can be justified under belief in the SES. The fact that all plans are extensive-form rationalizable implies that when believing in the SES is no more possible, players still ascribe to the opponents the highest level of strategic sophistication that it is compatible with the observed behavior.

[10] In games with more than two players or stages, a third condition is required: see Catonini (2019b).

[11] The original notion of Self-Justifiability of Catonini (2019b) requires μ_i to strongly believe also S_{-i}^∞, but this can be shown to be superfluous in 2-players games.

3 Solution

From now on, when not clear from the context, I will denote with $p_i^*(a)$ and $\overline{p}_i(a)$ the values of p_i^* and $\overline{p}_i$ in the subgame with locations $a \in [0,1]^2$.

3.1 Existence

In this section, I show that there exists a SES that prescribes locations $(a_1, a_2) = (1/4, 3/4)$, and the pure pricing equilibrium $(p_i^*)_{i=1,2} = (1,1)$ thereafter. For each $i = 1, 2$, let S_i^z be the set of plans $s_i \in S_i(a_i)$ with $s_i(a_{-i}) = 1$. Let:

$$S_1^{es} \;:\; = \left\{ s_1 \in S_1^\infty \cap S_1^z \;\middle|\; \begin{array}{c} \forall a_2' \in (1/4, 3/4) \cup (3/4, 1]\,, s_1(a_2') < \overline{p}_1(1/4, a_2') \\ \forall a_2' \in [0, 1/4)\,, s_1(a_2') \leq 3/4 - a_2' \end{array} \right\},$$

$$S_2^{es} \;:\; = \left\{ s_2 \in S_2^\infty \cap S_2^z \;\middle|\; \begin{array}{c} \forall a_1' \in [0, 1/4) \cup (1/4, 3/4)\,, s_2(a_1') < \overline{p}_2(a_1', 3/4) \\ \forall a_1' \in (3/4, 1]\,, s_2(a_1') \leq a_1' - 1/4 \end{array} \right\}.$$

As anticipated in the introduction, each S_i^{es} prescribes prices that a competitor who relocates towards the middle would not have the incentive to undercut (i.e., below $\overline{p}_i$).

I will show that each S_i^{es} is non-empty; now, I show that $S^{es} = S_1^{es} \times S_2^{es}$ satisfies Realization-strictness and Self-Justifiability, focusing without loss of generality on firm 1. For each $a_1' < 3/4$, at $(a_1', 3/4)$, the profit of firm 1 against any $p_2 < \overline{p}_2$ is strictly lower than the supremum of the profit against $\overline{p}_2$:

$$\overline{p}_2 - \left(\frac{3}{4} - a_1'\right) = \frac{1}{2},$$

For each $a_1' \in (3/4, 1]$, at $(a_1', 3/4)$, the profit of firm 1 against $p_2 \leq a_1' - 1/4$ is strictly lower than

$$a_1' - \frac{1}{4} - \left(a_1' - \frac{3}{4}\right) = \frac{1}{2}.$$

The equilibrium profit at $(1/4, 3/4)$ is exactly $1/2$. Therefore, for every $\mu_1 \in \Delta^H(S_2^{es})$ and $s_1 \in \rho_1(\mu_1)$, $s_1^\emptyset = 1/4$ and $s_1(3/4) = 1$, establishing Realization-strictness. Moreover, for every $s_1 \in S_1^{es} \subset S_1^1$, there is μ_1' with $s_1 \in \rho_1(\mu_1')$;

then, we have $s_1 \in \rho_1(\mu_1)$ for any $\mu_1 \in \Delta^H(S_2^{es})$ with $\mu_1(\cdot|1/4, a_2') = \mu_1'(\cdot|1/4, a_2')$ for each $a_2' \neq 3/4$, establishing Self-Justifiability.

Now I show that $S^{es} \neq \emptyset$ *in a fine discretization of the model.* To save on notation, instead of formally introducing a discretized model, I am going to use the original description of the model with the understanding that only rich but finite subsets of prices (and locations) are available. The proof for the continuous model, deferred to Appendix A, follows a different strategy, which also shows that all locations are compatible with extensive-form rationalizability.

In a discrete model, $S^\infty = S^m$ for some m. So, suppose by induction that

$$S_{1,n}^{es} \ : \ = \left\{ s_1 \in S_1^n \cap S_1^z \,\middle|\, \begin{array}{c} \forall a_2' \in (1/4, 3/4) \cup (3/4, 1]\,, s_1(a_2') < \overline{p}_1(1/4, a_2') \\ \forall a_2' \in [0, 1/4)\,, s_1(a_2') \leq 3/4 - a_2' \end{array} \right\} \neq \emptyset,$$

$$S_{2,n}^{es} \ : \ = \left\{ s_2 \in S_2^n \cap S_2^z \,\middle|\, \begin{array}{c} \forall a_1' \in [0, 1/4) \cup (1/4, 3/4)\,, s_2(a_1') < \overline{p}_2(a_1', 3/4) \\ \forall a_1' \in (3/4, 1]\,, s_2(a_1') \leq a_1' - 1/4 \end{array} \right\} \neq \emptyset.$$

Without loss of generality, focus on firm 1. By the arguments used for Realization-strictness, for each $\mu_1 \in \Delta^H(S_{2,n}^{es}) \cap \Delta^H(S_2^n)$, we have $\rho_1(\mu_1) \subseteq S_1^z \cap S_1^{n+1} \neq \emptyset$. Thus, for each $a_2' \neq 3/4$, we can define

$$p_1^{a_2'} := \min \left\{ p_1 \,\middle|\, \exists s_1 \in S_1^z \cap S_1^{n+1}, s_1(a_2') = p_1 \right\}$$

and fix $\mu_1^{a_2'} \in \Delta^H(S_2^n)$ such that $s_1'(a_2') = p_1^{a_2'}$ for some $s_1' \in \rho_1(\mu_1^{a_2'})$. Fix $\mu_1 \in \Delta^H(S_{2,n}^{es}) \cap \Delta^H(S_2^n)$ and $s_1 \in \rho_1(\mu_1) \subseteq S_1^z \cap S_1^{n+1}$ such that, for each $a_2' \neq 3/4$, $\mu_1(\cdot|1/4, a_2') = \mu_1^{a_2'}(\cdot|1/4, a_2')$ and $s_1(a_2') = p_1^{a_2'}$ (μ_1 exists because $S_2^n \neq \emptyset$ by the induction hypothesis and because $\mu_1(S_2(a_2')|\emptyset) = 0$). Suppose by contradiction that $S_{1,n+1}^{es} = \emptyset$, so $s_1 \notin S_{1,n+1}^{es}$. So, there is $a_2' \neq 3/4$ such that $s_1(a_2') > 3/4 - a_2'$ if $a_2' < 1/4$, $s_1(a_2') \geq \overline{p}_1(1/4, a_2')$ if $a_2' > 1/4$. Then, for every $\mu_2 \in \Delta(S_1^{n+1})$ with $\mu_2(s_1|\emptyset) = 1$, by the comparisons of profits, there is $s_2 \in \rho_2(\mu_2) \cap S_2(a_2') \subset S_2^{n+2}$ for some $a_2' \neq 3/4$. Suppose for simplicity that $\overline{p}_1(1/4, a_2')$ is not available in the discretization, thus $s_1(a_2') \neq \overline{p}_1(1/4, a_2')$.[12] Then, the best reply $s_2(1/4)$ to $s_1(a_2')$ is below $s_1(a_2') - |a_2' - 1/4|$, and in turn

[12] This is only to reduce the number of passages that lead to contradiction.

the best reply to $s_2(1/4)$ is below $s_1(a_2')$. Hence, fix $\mu_1' \in \Delta^H(S_{2,n}^{es}) \cap \Delta^H(S_2^n)$ with $\mu_1'(s_2|1/4, a_2') = 1$; there is $s_1' \in \rho_1(\mu_1') \subseteq S_1^z \cap S_1^{n+1}$ such that $s_1'(a_2') < s_1(a_2') = p_1^{a_2'}$, contradicting the definition of $p_1^{a_2'}$.

A final comment. The proof of $S^{es} \neq \emptyset$ relies only on beliefs where each firm i gives probability 1 to plans in S_{-i}^z. This means that the threats that sustain the SES are compatible also with a special kind of forward induction reasoning where deviations are always interpreted as attempts to get a higher payoff than under the agreed-upon path (as opposed to disbelief in the agreed-upon path). Formally, the SES exists also under "epistemic priority to the path" — a condition introduced by Catonini (2019b) that refines SES's. In the equilibrium literature, instances of this kind of forward induction reasoning are captured by strategic stability (Kohlberg and Mertens, 1986) and related refinement — see the Intuitive Criterion (Cho and Kreps, 1987) as (arguably) the most transparent example.

3.2 Uniqueness

The goal of this section is to show that there is no symmetric SES that prescribes a locations pair different than $(1/4, 3/4)$. When clear from the context, I will denote with P_i^n the set $P_i^n(a)$ of prices of firm i that survive n steps of rationalizability in the subgame with locations a (see Section 2.3).

Note preliminarily two facts. First, a SES $S_1^{es} \times S_2^{es} \subseteq S_1(a_1) \times S_2(a_2)$, at (a_1, a_2), must prescribe all and only the prices that best reply to some conjecture over the prices prescribed to the competitor; that is, a set of prices *closed under rational behavior* (Basu and Weibull 1991). This is obvious from Realization-strictness and Self-Justifiability.

Second, at each locations pair (a_1, a_2) with $a_1 < 1/2 < a_2$, for each $i = 1, 2$, no price above p_i^* is rationalizable. Note preliminarily that, by $a_1 < 1/2 < a_2$,

$$|p_1^* - p_2^*| = \frac{2}{3}|a_2 - (1 - a_1)| < a_2 - a_1. \tag{9}$$

Since demand vanishes for sufficiently high prices, each P_i^1 is bounded. So, for each $n > 0$ and $i = 1, 2$, let $p_i^{S,n} := \sup P_i^n$. First, note that no $p_i > p_i' := \max\left\{p_{-i}^{S,n} - (a_2 - a_1), p_i^F(p_{-i}^{S,n})\right\}$ is a best reply to a conjecture over P_{-i}^n: against any $p_{-i} \in [0, p_{-i}^{S,n}]$, firm i's profit is zero or strictly decreasing above p_i'. Thus, $p_i^{S,n+1} \leq p_i'$. If $p_{-i}^{S,n} \leq p_{-i}^*$, $p_i^F(p_{-i}^{S,n}) \leq p_i^*$, and by (9), $p_{-i}^{S,n} - (a_2 - a_1) < p_i^*$. Hence, $p_i^* \geq p_i' \geq p_i^{S,n+1}$. If $p_i^{S,n} > p_i^*$ for every $i = 1, 2$, proceed as follows. Write $p_i^{S,n} = p_i^* + x_i$; then,

$$
\begin{aligned}
p_1^F(p_2^{S,n}) &= p_1^F(p_2^* + x_2) = \frac{a_1 + a_2}{2} + \frac{4 - a_1 - a_2}{6} + \frac{1}{2}x_2 = \\
&= \frac{2 + a_1 + a_2}{3} + \frac{1}{2}x_2 = p_1^* + \frac{1}{2}x_2; \\
p_2^F(p_1^{S,n}) &= p_2^* + \frac{1}{2}x_1.
\end{aligned}
$$

Each $p_i^{S,n+1} - p_i^*$ is bounded above by $p_i' - p_i^*$, thus by the maximum between $p_i^F(p_{-i}^{S,n}) - p_i^* = x_{-i}/2$ and $p_{-i}^{S,n} - (a_2 - a_1) - p_i^* = x_{-i} - (a_2 - a_1) - \left(p_i^* - p_{-i}^*\right)$. Hence, $\max\left\{p_1^{S,n+1} - p_1^*, p_2^{S,n+1} - p_2^*\right\}$ is bounded above by

$$
\max\left\{\max\left\{x_1, x_2\right\}/2, \max\left\{x_1, x_2\right\} - (a_2 - a_1) + |p_1^* - p_2^*|\right\}.
$$

Since $|p_1^* - p_2^*| - (a_2 - a_1)$ is negative by (9) and does not depend on x_1 and x_2, every $p_i > p_i^*$ is eventually eliminated.

3.2.1 Farther from the center

Let $a_1 < 1/4$ and $a_2 = 1 - a_1$, and suppose by contradiction that there exists a SES $S_1^{es} \times S_2^{es} \subset S_1(a_1) \times S_2(a_2)$. I will show that for each $a_1' \in [0, 1/4)$, (p_1^*, p_2^*) is the only rationalizable price pair at (a_1', a_2), hence the only price set closed under rational behavior. Thus, for every $(s_1, s_2) \in S_1^{es} \times S_2^{es}$, $s_1(a_2) = s_2(a_1) = 1$. I will also show that then, for all $n \geq 0$ and $a_1' \in (a_1, 1/4)$, $S_1^n \cap S_1(a_1') \neq \emptyset$, thus $S^n \cap (S_1(a_1') \times S_2(a_2)) \neq \emptyset$. As observed in Section 2.3, this implies that for each $s_2 \in S_2^\infty \cap S_2(a_2) \supset S_2^{es}$, $s_2(a_1')$ is a rationalizable price. Thus, $s_2(a_1') = p_2^*(a_1', a_2)$. Hence, the profit of firm 1 at (a_1', a_2) against

s_2 is

$$\frac{1}{2}\left(p_1^*(a_1',a_2)\right)^2 = \frac{1}{2}\left(\frac{2+a_1'+a_2}{3}\right)^2 > \frac{1}{2}. \tag{10}$$

Therefore, for every $\mu_1 \in \Delta^H(S_2^{es})$, we have $\rho_1(\mu_1) \cap S_1(a_1) = \emptyset$, violating Realization-Strictness.

Now I show that, for each $a_1' \in (a_1, 1/4)$, $S_1^n \cap S_1(a_1') \neq \emptyset$ for all n. Fix $s_2 \in S_2^{es}$ and define $s_2' \in S_2(a_2)$ as $s_2'(a_1') = p_2^*(a_1', a_2)$ and $s_2'(a_1'') = s_2(a_1'')$ for each $a_1'' \neq a_1'$. Suppose by way of induction that $s_2' \in S_2^n$, and that there is $s_1 \in S_1(a_1') \cap S_1^n$ with $s_1(a_2) = p_1^*(a_1', a_2)$.

By Self-Justifiability, there is $\mu_2 \in \Delta^H(S_1^{es})$ such that $s_2 \in \rho_2(\mu_2)$; by $s_2 \in S_2^\infty$, there is $\widetilde{\mu}_2 \in \Delta^H(S_1^n)$ such that $s_2 \in \rho_2(\widetilde{\mu}_2)$. Since $\mu_2(S_1^\infty \cap S_1(a_1)|\emptyset) = 1$, I can construct $\mu_2' \in \Delta^H(S_1^n)$ such that $\mu_2'(\cdot|\emptyset) = \mu_2(\cdot|\emptyset)$, $\mu_2'(s_1|a_1', a_2) = 1$, and $\mu_2'(\cdot|a_1'', a_2) = \widetilde{\mu}_2(\cdot|a_1'', a_2)$ for all $a_1'' \notin \{a_1, a_1'\}$. Thus, $s_2' \in \rho_2(\mu_2') \subset S_2^{n+1}$.

Fix $\mu_1 \in \Delta^H(S_2^{es})$ with $\mu_1(s_2|\emptyset) = 1$, and $\mu_1' \in \Delta^H(S_2^n)$ with $\mu_1'(s_2'|\emptyset) = 1$. By Realization-strictness, $\rho_1(\mu_1) \subseteq S_1(a_1)$. But then, since s_2 and s_2' prescribe different prices only at (a_1', a_2), we have $\rho_1(\mu_1') \subseteq S_1(a_1) \cap S_1(a_1')$. So, since firm 1's profit against s_2' is higher at (a_1', a_2) than at (a_1, a_2) by (10), for any $s_1' \in \rho_1(\mu_1') \subset S_1^{n+1}$, $s_1' \in S_1(a_1')$ (and $s_1'(a_2) = p_1^*(a_1', a_2)$).

Finally, I prove that, at every (a_1, a_2) with $a_1 < 1/4$ and $a_2 > 3/4$, the pure equilibrium is also the only rationalizable price pair.[13] I have already shown that, for each $i = 1, 2$, no price above p_i^* is rationalizable. Now I show that no price below p_i^* is rationalizable. For each $n > 0$ and $i = 1, 2$, let $p_i^{I,n} := \inf P_i^n$.

First, I show that, for each $i = 1, 2$, there exists $\varepsilon > 0$ such that, for every $\widetilde{p} \leq \max\{p_1^*, p_2^*\} - (a_2 - a_1)$, each $p_i \in [\widetilde{p}, \widetilde{p} + \varepsilon]$ is dominated over $[\widetilde{p}, \infty)$, by $2p_i$. Then, in a finite number of steps m, we obtain $p_i^{I,m} > \max\{p_1^*, p_2^*\} - (a_2 - a_1)$.

[13] This is the reason why $(1/4, 3/4)$ can be induced by a coarser agreement than the SES of Section 3.1: the firms can simply agree to keep the interval $(1/4, 3/4)$ as a buffer zone, and threaten to fix a sufficiently low price if the competitor violates it. With strategic reasoning, each firm, say firm 1, would progressively realize that unilaterally relocating from 0 towards 1/4 guarantees a higher price, other than higher demand. The self-enforceability of such agreement must be verified with a refinement of extensive-form rationalizability with first-order-belief restrictions, called Selective Rationalizability, introduced by Catonini (2019a).

Without loss of generality, focus on firm 1. Note that

$$2\widetilde{p} \leq \widetilde{p}+\max\{p_1^*,p_2^*\}-(a_2-a_1)=\widetilde{p}+1+\frac{|a_1+a_2-1|}{3}-(a_2-a_1)<\widetilde{p}+(a_2-a_1), \quad (11)$$

$$2\widetilde{p} \leq 2\max\{p_1^*,p_2^*\}-2(a_2-a_1)=2+\frac{2\,|a_1+a_2-1|}{3}-2(a_2-a_1)<a_1+a_2, \quad (12)$$

where the strict inequalities are equalities for $(a_1,a_2)=(1/4,3/4)$, thus satisfied for $a_1<1/4$ and $a_2>3/4$. Fix first $p_2\in[\widetilde{p},\widetilde{p}+(a_2-a_1))$. Then, the profits of firm 1 with $p_1=\widetilde{p}$ and $p_1=2\widetilde{p}$ are, respectively $\pi_1(\widetilde{p},p_2)$ and, by (11), $\pi_1(2\widetilde{p},p_2)$. We have

$$\begin{aligned}\pi_1(\widetilde{p},p_2)-\pi_1(2\widetilde{p},p_2) &= \widetilde{p}\left(\frac{a_1+a_2}{2}+\frac{p_2-\widetilde{p}}{2}\right)-2\widetilde{p}\left(\frac{a_1+a_2}{2}+\frac{p_2-2\widetilde{p}}{2}\right)\\ &= \frac{1}{2}\widetilde{p}\,(3\widetilde{p}-p_2-a_1-a_2)\leq\frac{1}{2}\widetilde{p}\,(2\widetilde{p}-a_1-a_2)<0\end{aligned}$$

where the strict inequality is due to (12). Now let $p_2\in[\widetilde{p}+(a_2-a_1),\infty)$. The profit of firm 1 with $p_1=\widetilde{p}$ is at most $\widetilde{p}$. The profit with $p_1=2\widetilde{p}$ is minimal at $p_2=\widetilde{p}+(a_2-a_1)$ and reads

$$\begin{aligned}\pi_1(2\widetilde{p},p_2) &= 2\widetilde{p}\left(\frac{a_1+a_2}{2}+\frac{\widetilde{p}+(a_2-a_1)-2\widetilde{p}}{2}\right)=\\ &= \widetilde{p}\,(2a_2-\widetilde{p})\geq\widetilde{p}\,(2a_2-\max\{p_1^*,p_2^*\}+(a_2-a_1))=\\ &= \widetilde{p}\left(3a_2-a_1-1-\frac{|a_1+a_2-1|}{3}\right)>\widetilde{p},\end{aligned}$$

where the strict inequality is an equality when $(a_1,a_2)=(1/4,3/4)$, thus satisfied for $a_1<1/4$ and $a_2>3/4$. Fix $\varepsilon>0$ such that, for $\widetilde{p}=\max\{p_1^*,p_2^*\}-(a_2-a_1)$, we preserve:

1. $2(\widetilde{p}+\varepsilon)<\widetilde{p}+(a_2-a_1)$;

2. for each $p_2\in[\widetilde{p},\widetilde{p}+(a_2-a_1))$,

$$\pi_1(\widetilde{p}+\varepsilon,p_2)-\pi_1(2(\widetilde{p}+\varepsilon),p_2)=\frac{1}{2}(\widetilde{p}+\varepsilon)\,(3(\widetilde{p}+\varepsilon)-p_2-a_1-a_2)<0;$$

3. for each $p_2 \in [\widetilde{p} + (a_2 - a_1), \infty)$,

$$\widetilde{p} + \varepsilon - \pi_1(2(\widetilde{p} + \varepsilon), p_2) = (\widetilde{p} + \varepsilon)\left(1 - (a_1 + a_2 + p_2 - 2(\widetilde{p} + \varepsilon))\right) < 0.$$

Note that all three inequalities hold also for $\widetilde{p} < \max\{p_1^*, p_2^*\} - (a_2 - a_1)$: the first is obvious, and if the left-hand sides of the second and third are negative for a given $\widetilde{p}$, so they are for a lower one. Hence, we found the value of ε we were looking for.

So, there is m such that, for each $i = 1, 2$, since no $p_{-i} > p_{-i}^*$ is rationalizable, $p_i^{I,m} > p_{-i}^{S,m} - (a_2 - a_1)$. For each $p_{-i} \in [p_{-i}^{I,m}, p_{-i}^{S,m}]$, $\pi_i(\cdot, p_{-i})$ increases over $[p_i^{I,m}, p_i^F(p_{-i}))$,[14] where $p_i^F(p_{-i}^{I,m}) \le p_i^F(p_{-i}) \le p_{-i} + (a_2 - a_1)$ (we have $\underline{p}_{-i} < 0$). Hence, $p_i^F(p_{-i}^{I,m})$ dominates each $p_i \in [p_i^{I,m}, p_i^F(p_{-i}^{I,m}))$ over $[p_{-i}^{I,m}, p_{-i}^{S,m}]$. Thus, for each $i = 1, 2$, $p_i^{I,m+1} \ge p_i^F(p_{-i}^{I,m})$. Then, it is easy to see by induction that each $p_i < p_i^*$ is eliminated at some step.

3.2.2 Closer to the center

Let $a_1 \in (1/4, 1/2)$ and $a_2 = 1 - a_1$, and suppose by contradiction that there exists a SES $S_1^{es} \times S_2^{es} \subseteq S_1(a_1) \times S_2(a_2)$. For each $i = 1, 2$, let P_i^{es} be the set of prices prescribed by S_i^{es} at (a_1, a_2). I am going to find a deviation $a_1' < a_1$ such that firm 1's profit at (a_1, a_2) against some $p_2 \in P_2^{es}$ is strictly lower than at (a_1', a_2) against any rationalizable price of firm 2. So, there is $\overline{s}_2 \in S_2^{es}$ such that firm 1's profit against $\overline{s}_2(a_1)$ at (a_1, a_2) is strictly lower than against any $p_2 \in P_2^{\infty}$ at (a_1', a_2). With this, I show that, for all $n > 0$, $S_1^n \cap S_1(a_1') \neq \emptyset$, thus $S^n \cap (S_1(a_1') \times S_2(a_2)) \neq \emptyset$. As observed in Section 2.3, this implies that $\overline{s}_2(a_1')$ is rationalizable. Then, fixing $\mu_1 \in \Delta^H(S_2^{es})$ with $\mu_1(\overline{s}_2|\emptyset) = 1$, we have $\rho_1(\mu_1) \cap S_1(a_1) = \emptyset$, contradicting Realization-strictness.

I start from showing that $S_1^n \cap S_1(a_1') \neq \emptyset$ for all $n > 0$. Let $P_1 \times P_2 \subset P_1^{\infty} \times P_2^{\infty}$ be a best response set of prices at (a_1', a_2) — see Appendix A.1 for

[14] I am not assuming that $p_i^F(p_{-i}^{I,m}) > p_i^{I,m}$, thus that $[p_i^{I,m}, p_i^F(p_{-i}^{I,m}))$ is non-empty. If it was for one firm, it would not be for the other, because for $p_{-i}^{I,m} < p_{-i}^*$, $p_{-i}^F(p_i^F(p_{-i}^{I,m})) > p_{-i}^{I,m}$. Hence, the inductive eliminations never stop.

its existence in the continuous model. Let $\widehat{S}_2$ be the set of all $s_2 \in S_2(a_2)$ such that $s_2(a_1') \in P_2$ and $s_2(a_1'') = \overline{s}_2(a_1'')$ for each $a_1'' \neq a_1'$. Fix $n \geq 0$ and suppose by way of induction that $\widehat{S}_2 \subseteq S_2^n$, and that for each $p_1 \in P_1$, there is $s_1 \in S_1(a_1') \cap S_1^n$ such that $s_1(a_2) = p_1$.

By Self-Justifiability, there is $\mu_2 \in \Delta^H(S_1^{es})$ such that $\overline{s}_2 \in \rho_2(\mu_2)$; by $\overline{s}_2 \in S_2^\infty$, there is $\widetilde{\mu}_2 \in \Delta^H(S_1^n)$ such that $\overline{s}_2 \in \rho_2(\widetilde{\mu}_2)$. By $\mu_2(S_1^\infty \cap S_1(a_1)|\emptyset) = 1$ and the induction hypothesis, for each $s_2 \in \widehat{S}_2$ I can construct $\mu_2' \in \Delta^H(S_1^n)$ with $\mu_2'(\cdot|\emptyset) = \mu_2(\cdot|\emptyset)$ such that $\mu_2'(\cdot|a_1', a_2)$ justifies $s_2(a_1')$ and $\mu_2'(\cdot|a_1'', a_2) = \widetilde{\mu}_2(\cdot|a_1'', a_2)$ for all $a_1'' \notin \{a_1, a_1'\}$. Thus, $s_2 \in \rho_2(\mu_2') \subseteq S_2^{n+1}$.

Fix $p_1 \in P_1$. Fix $\mu_1 \in \Delta^H(S_2^{es})$ with $\mu_1(\overline{s}_2|\emptyset) = 1$, and $\mu_1' \in \Delta^H(S_2^n) \cap \Delta^H(\widehat{S}_2)$ such that $\mu_1'(\cdot|a_1', a_2)$ induces a distribution over prices that justifies p_1 (μ_1' exists by the induction hypothesis). By Realization-strictness, $\rho_1(\mu_1) \subseteq S_1(a_1)$. But then, since $\mu_1(\cdot|\emptyset)$ and $\mu_1'(\cdot|\emptyset)$ induce different distributions over prices only at (a_1', a_2), we have $\rho_1(\mu_1') \subseteq S_1(a_1) \cap S_1(a_1')$. Hence, by $\mu_1'(\widehat{S}_2|\emptyset) = 1$, $\rho_1(\mu_1') \subseteq S_1(a_1')$, and there is $s_1 \in \rho_1(\mu_1') \subset S_1^{n+1}$ with $s_1(a_2) = p_1$.

For most values of a_1, we can pick $a_1' = 0$. Since $a_1' = 0$ facilitates exposition, I will adopt it whenever it works, and use an alternative a_1' otherwise.

1. Let $\overline{a}_1 \simeq 0.34$ be the value of a_1 such that, at (a_1, a_2), $\underline{p}_2 = \overline{p}_2 - (a_2 - a_1)$. Thus, at (a_1, a_2) with $a_1 \in (1/4, \overline{a}_1)$, $\overline{p}_2 - (a_2 - a_1) > \underline{p}_2$.

2. Let $\overline{\overline{a}}_1 \simeq 0.38$ be the value of a_1 such that, at (a_1, a_2), $\underline{p}_2 = \overline{p}_2$. Hence, $\overline{p}_2 - (a_2 - a_1) \leq \underline{p}_2 < \overline{p}_2$ if $a_1 \in [\overline{a}_1, \overline{\overline{a}}_1)$, $\overline{p}_2 \leq \underline{p}_2$ if $a_1 \in [\overline{\overline{a}}_1, 1/2)$.

3. Let $\overline{\overline{a}}_1' \simeq 0.42$ be the value of a_1 such that, at $(0, a_2) = (0, 1 - a_1)$, $\underline{p}_1 = \overline{p}_2 - a_2$. Thus, at $(0, a_2)$, $\overline{p}_2 - a_2 > \underline{p}_1$ if $a_1 \in (1/4, \overline{\overline{a}}_1')$, $\overline{p}_2 - a_2 \leq \underline{p}_1$ if $a_1 \in [\overline{\overline{a}}_1', 1/2)$.

4. Let $\overline{a}_1' \simeq 0.37$ be the value of a_1 such that $p_1^F(\underline{p}_2)$ at (a_1, a_2) is equal to $p_1^F(p_2^F(\overline{p}_2 - a_2))$ at $(0, a_2)$.

Let $\widetilde{a}_1' \simeq 0.15$ be the value of $\widetilde{a}_1$ such that, at $(\widetilde{a}_1, a_2) = (\widetilde{a}_1, 1 - \overline{\overline{a}}_1)$, $\underline{p}_1 = \overline{p}_2 - (a_2 - \widetilde{a}_1)$ ($\widetilde{a}_1'$ solves (13)). Thus, at $(\widetilde{a}_1', a_2) = (\widetilde{a}_1', 1 - a_1)$, $\overline{p}_2 - (a_2 - \widetilde{a}_1') > \underline{p}_1$ if $a_1 < \overline{\overline{a}}_1$, $\overline{p}_2 - (a_2 - \widetilde{a}_1') \leq \underline{p}_1$ if $a_1 \geq \overline{\overline{a}}_1$.

I will let p^U denote an upper bound for $\inf P_2^{es}$, and π_1 an upper bound of firm 1's profit against p^U. I will let p^L denote a lower bound for $\inf P_2^{\infty}$ at (a_1', a_2), and π_1' firm 1's *optimal* profit against p^L. I will determine and compare the following values:

a_1	$(1/4, \overline{a}_1)$	$[\overline{a}_1, \overline{a}_1')$	$[\overline{a}_1', \overline{\overline{a}}_1)$	$[\overline{\overline{a}}_1, \overline{\overline{a}}_1']$	$(\overline{\overline{a}}_1', 1/2)$
p^U	$\overline{p}_2 - (a_2 - a_1)$	$\underline{p}_2$	$\underline{p}_2$	$\overline{p}_2$	$\overline{p}_2$
π_1	$\left(p_1^F(p^U)\right)^2/2$	$\left(p_1^F(p^U)\right)^2/2$	$\left(p_1^F(p^U)\right)^2/2$	$\overline{p}_2 - (a_2 - a_1)$	$\overline{p}_2 - (a_2 - a_1)$
a_1'	0	0	$\widetilde{a}_1'$	$\widetilde{a}_1'$	0
p^L	$p_2^F(\overline{p}_2 - a_2)$	$p_2^F(\overline{p}_2 - a_2)$	$p_2^F(\overline{p}_2 - (a_2 - a_1'))$	$\overline{p}_2$	$\overline{p}_2$
π_1'	$\left(p_1^F(p^L)\right)^2/2$	$\left(p_1^F(p^L)\right)^2/2$	$\left(p_1^F(p^L)\right)^2/2$	$\left(p_1^F(\overline{p}_2)\right)^2/2$	$\left(p_1^F(\overline{p}_2)\right)^2/2$

Profits First, suppose that $a_1 \in [\overline{\overline{a}}_1, 1/2)$. At (a_1, a_2), by $a_1 \geq \overline{\overline{a}}_1$, $\overline{p}_2 \leq \underline{p}_2$, thus firm 1's profit against $p^U = \overline{p}_2$ is strictly below $\pi_1 = \overline{p}_2 - (a_2 - a_1) = (p_1^F(\overline{p}_2))^2/2$. At (a_1', a_2), $\underline{p}_2 < 0$, therefore firm 1's profit against $p^L = \overline{p}_2$ is exactly $\pi_1' = (p_1^F(\overline{p}_2))^2/2$. I will show later that there is $p_2 \in P_2^{es}$ with $p_2 \leq p^U$. Since $p_1^F(\overline{p}_2)$ is independent of firm 1's location, firm 1's profit against p_2 at (a_1, a_2) is strictly lower than against any $p_2' \in P_2^{\infty}$ at (a_1', a_2), as desired.

Second, suppose that $a_1 \in (1/4, \overline{\overline{a}}_1)$. At (a_1, a_2), since $p^U < \overline{p}_2$, $\pi_1 = \left(p_1^F(p^U)\right)^2/2$ is an upper bound of firm 1's profit against p^U.[15] At (a_1', a_2), we have $\overline{p}_2 - (a_2 - a_1') > \underline{p}_1$, by $a_1 < \overline{\overline{a}}_1'$ and $a_1' = 0$ for $a_1 \in (1/4, \overline{a}_1')$, by $a_1 < \overline{\overline{a}}_1$ and $a_1' = \widetilde{a}_1'$ for $a_1 \in [\overline{a}_1', \overline{\overline{a}}_1)$. Then, $p^L = p_2^F(\overline{p}_2 - (a_2 - a_1')) < \overline{p}_2$. Moreover, since firm 2 is closer to the center, $\underline{p}_2 < \underline{p}_1$ and $p^L > \overline{p}_2 - (a_2 - a_1')$. Hence, $\underline{p}_2 < p^L < \overline{p}_2$. Thus, $\pi_1' = \left(p_1^F(p^L)\right)^2/2$ is firm 1's optimal profit against p^L.

Now I want to show that $\pi_1' > \pi_1$; then, there is $\varepsilon > 0$ such that for any $p_2 \in \left[0, p^U + \varepsilon\right]$ (thus for some $p_2 \in P_2^{es}$), firm 1's profit against p_2 at (a_1, a_2) is smaller than against any $p_2' \in P_2^{\infty}$ at (a_1', a_2), as desired.

For $a_1 \in (1/4, \overline{a}_1)$, at (a_1, a_2), we have

$$p_1^F(p^U) = \frac{1}{2} + \frac{1}{2}(4 - 2a_2 - 4\sqrt{1 - a_2}) =: p_1',$$

[15] It is actually the maximum for $a_1 \in (1/4, \overline{a}_1)$ and the supremum for $[\overline{a}_1, \overline{\overline{a}}_1)$.

and at $(a_1', a_2) = (0, 1-a_1)$ we have

$$p_1^F(p^L) = \frac{1}{2}a_2 + \frac{1}{2}\left(1 - \frac{1}{2}a_2 + \frac{1}{2}\left(4 - 2a_2 - 4\sqrt{1-a_2}\right)\right) =: p_1''.$$

We obtain $\pi_1' > \pi_1$ because

$$p_1'' > p_1' \Leftrightarrow \frac{3}{4}a_2 + \sqrt{1-a_2} > 1 \Leftrightarrow 4\sqrt{a_1} - 1 - 3a_1 > 0 \Leftrightarrow a_1 \in \left(\frac{1}{9}, 1\right).$$

At $(\overline{a}_1', 1-\overline{a}_1')$, we have $p_1^F(\underline{p}_2) = 2\overline{a}_1'$; at $(0, 1-\overline{a}_1')$,

$$p_1^F(p_2^F(\overline{p}_2 - a_2)) = \frac{5}{4} + \frac{1}{4}\overline{a}_1' - \sqrt{\overline{a}_1'}.$$

So, by definition of $\overline{a}_1'$,

$$2\overline{a}_1' = \frac{5}{4} + \frac{1}{4}\overline{a}_1' - \sqrt{\overline{a}_1'}. \tag{13}$$

For $a_1 \in [\overline{a}_1, \overline{a}_1')$, with a_1 in place of $\overline{a}_1'$, the left-hand side of (13) is $p_1^F(p^U)$ at (a_1, a_2) and it is increasing, while the right-hand side is $p_1^F(p^L)$ at $(0, a_2)$ and it is decreasing. Thus, $p_1^F(p^U)$ at (a_1, a_2) is lower than $p_1^F(p^L)$ at $(0, a_2)$. So, $\pi_1' > \pi_1$.

By definition of $\overline{\overline{a}}_1$, at $(\overline{\overline{a}}_1, 1-\overline{\overline{a}}_1)$ we have $\overline{p}_2 = \underline{p}_2$, hence $p_1^F(\overline{p}_2) = 2\overline{\overline{a}}_1$. By definition of $\widetilde{a}_1'$, at $(\widetilde{a}_1', 1-\overline{\overline{a}}_1)$ we have $\overline{p}_2 = p_2^F(\overline{p}_2 - (a_2 - \widetilde{a}_1'))$, thus

$$p_1^F(\overline{p}_2) = p_1^F(p_2^F(\overline{p}_2 - (a_2 - \widetilde{a}_1'))) = \frac{5}{4} + \frac{1}{4}\overline{\overline{a}}_1 + \frac{1}{4}\widetilde{a}_1' - \sqrt{\overline{\overline{a}}_1}. \tag{14}$$

Since $p_1^F(\overline{p}_2)$ is independent of firm 1's location,[16] we obtain

$$2\overline{\overline{a}}_1 = \frac{5}{4} + \frac{1}{4}\overline{\overline{a}}_1 + \frac{1}{4}\widetilde{a}_1' - \sqrt{\overline{\overline{a}}_1}. \tag{15}$$

For $a_1 \in [\overline{a}_1', \overline{\overline{a}}_1)$, with a_1 in place of $\overline{\overline{a}}_1$, the left-hand side of (15) is $p_1^F(p^U)$ at

[16] Indeed, equation 14 is false if $\widetilde{a}_1'$ is substituted with another location of firm 1.

(a_1, a_2) and it is increasing, while the right-hand side is $p_1^F(p^L)$ at $(\widetilde{a}_1', a_2)$ and it is decreasing. Hence, $\pi_1' > \pi_1$.

Determination of p^U Recall that $P_1^{es} \times P_2^{es}$ must be closed under rational behavior. By symmetry, $P_1^{es} = P_2^{es} =: P$, and let $p^I := \inf P$.[17] Let $\overline{p} := \overline{p}_1 = \overline{p}_2$, $\underline{p} := \underline{p}_1 = \underline{p}_2$, and $p^F(\cdot) := p_1^F(\cdot) = p_2^F(\cdot)$.

First I show that there is $p \in P$ with $p \leq \overline{p}$. Suppose not. Then, when a firm fixes some $p \in P$ close to p^I, the competitor has a best reply close to $p^I - (a_2 - a_1) < p^I$ in any fine discretization of the model, a contradiction. The argument for the continuous model is presented in Appendix B.

Now, I show that if $a_1 \in [\overline{a}_1, \overline{\overline{a}}_1)$, $p^I \leq \underline{p}$. Suppose not. Then, $\underline{p} < p^I \leq \overline{p}$. So, when a firm fixes $p \in P$ with $p \leq \overline{p}$ equal to or just above p^I, the competitor's best reply is $p^F(p) > p^F(\underline{p}) \geq \overline{p}$,[18] with $p^F(p) < p + (a_2 - a_1)$. But then, the first firm has a best reply to $p^F(p)$ just below $p^F(p) - (a_2 - a_1) < p^I$ in any fine discretization of the model, a contradiction. In the continuous model, take conjectures $\delta p^F(p) + (1-\delta)p$ for small δ's, so to generate a neighbourhood of best replies above $p^F(p) > \overline{p}$. A uniform distribution over a sufficiently small neighbourhood has best reply $p^F(p) - (a_2 - a_1)$ (see Appendix A.1).

Finally, I show that if $a_1 \in (1/4, \overline{a}_1)$, $p^I \leq \overline{p} - (a_2 - a_1)$. Suppose not. Then, $\underline{p} < \overline{p} - (a_2 - a_1) < p^I \leq \overline{p}$. If $p^F(p^I) > \overline{p}$, the same argument for the $a_1 \in [\overline{a}_1, \overline{\overline{a}}_1)$ case applies. Else, iteratively apply $p^F(\cdot)$ until reaching $p' \in P$ and $p'' = p^F(p') \in P$ such that $p'' \leq \overline{p} < p^F(p'')$. Fix $p''' \in (\overline{p}, p^F(p''))$ such that $p''' - (a_2 - a_1) < p^I$. Fix a distribution ν over $\{p', p''\}$ with mean $(p^F)^{-1}(p''')$.[19] In a fine discretization of the model, the best reply to ν is either below $p'' - (a_2 - a_1) \leq \overline{p} - (a_2 - a_1) < p^I$, a contradiction, or it is p''', but the best reply to p''' is below p^I, a contradiction as well. In the continuous model, the perturbation argument used for the $a_1 \in [\overline{a}_1, \overline{\overline{a}}_1)$ case applies.

[17] Symmetry is not crucial but it is maintaned to simplify the argument.

[18] $p^F(\underline{p}) \geq \overline{p}$ is due to $p^F(\underline{p}) = \underline{p} + (a_2 - a_1)$ and $\underline{p} \geq \overline{p} - (a_2 - a_1)$ by $a_1 \in [\overline{a}_1, \overline{\overline{a}}_1)$.

[19] Here symmetry is used to claim that the same firm can fix p' or p''. A more complex argument does away with symmetry.

Determination of p^L First, I show that for each $a_1 \in (1/4, 1/2)$, at $(a_1', a_2) = (0, 1 - a_1)$ there is no $p_2 \in P_2^5$ below $\min\left\{\overline{p}_2, p_2^F(\overline{p}_2 - a_2)\right\}$. Thus, for $a_1 \in (1/4, \overline{a}_1')$, at $(a_1', a_2) = (0, 1 - a_1)$ there is no $p_2 \in P_2^5$ below $p^L = p_2^F(\overline{p}_2 - a_2)$ (by $a_1 < \overline{\overline{a}}_1'$, $\overline{p}_2 - a_2 > \underline{p}_1$); for $a_1 \in (\overline{\overline{a}}_1', 1/4)$, at $(a_1', a_2) = (0, 1 - a_1)$ there is no $p_2 \in P_2^5$ below $p^L = \overline{p}_2$ (by $a_1 > \overline{\overline{a}}_1'$, $\overline{p}_2 - a_2 < \underline{p}_1$). Note preliminarily that $\underline{p}_2 < 0$, thus $p_1^F(p_2) < p_2 + (a_2 - a_1)$ for every p_2, and that by $a_1 = 0$ firm 2 has no incentive to undercut.

1. Every $p_2 < 1/2$ is dominated by any $p_2' \in (p_2, 1/2)$: for each $p_1 \geq 0$, firm 2's profit is strictly increasing over $\left[0, \min\left\{p_2^F(p_1), p_1 + a_2\right\}\right)$, and $\min\left\{p_2^F(0), a_2\right\} > 1/2$.

2. Every $p_1 < \min\{1/4, \overline{p}_2 - a_2\}$ is dominated by $p_1' := p_1^F(p_1 + a_2)$ over $[1/2, \infty)$. First, p_1' is the unique best reply to $p_2 = p_1 + a_2 < \overline{p}_2$, and for $p_2 > p_1 + a_2$ firm 1's profit with p_1' weakly increases, while with p_1 it remains p_1. Second, for $p_2 \in [1/2, p_1 + a_2)$, $p_1' = a_2 + p_1/2$ is closer than p_1 to $p_1^F(p_2) = (a_2 + p_2)/2 > a_2/2 + p_1$; so, since $\pi_1(\cdot, p_2)$ is a parabola, $\pi_1(p_1', p_2) > \pi_1(p_1, p_2)$ (and $\pi_1(p_1', p_2)$ is firm 1's profit under (p_1', p_2) because $p_2 - a_2 < p_1' < p_2 + a_2$).

If $\min\{1/4, \overline{p}_2 - a_2\} = \overline{p}_2 - a_2$, move to point 5; else, continue as follows.

3. Every $p_2 < 3/4$ is dominated by any $p_2' \in (p_2, 3/4)$ over $[1/4, \infty)$: for each $p_1 \geq 1/4$, firm 2's profit is strictly increasing over $\left[0, \min\left\{p_2^F(p_1), p_1 + a_2\right\}\right)$, and $\min\left\{p_2^F(1/4), a_2 + 1/4\right\} > 3/4$.

4. Every $p_1 < \overline{p}_2 - a_2 < 1/2$ is dominated by $p_1' := p_1^F(p_1 + a_2)$ over $[3/4, \infty)$: p_1' is the unique best reply to $p_2 = p_1 + a_2 < \overline{p}_2$, and for $p_2 > p_1 + a_2$ firm 1's profit with p_1' weakly increases, while with p_1 it remains p_1; for $p_2 \in [3/4, p_1 + a_2)$, $p_1' = a_2 + p_1/2 < p_2 + a_2$ is closer than p_1 to

$$p_1^F(p_2) = \frac{a_2 + p_2}{2} > \frac{a_2}{2} + \frac{3}{4}p_1 = \frac{p_1 + p_1'}{2}.$$

5. Every $p_2 < \min\left\{\overline{p}_2, p_2^F(\overline{p}_2 - a_2)\right\} =: p$ is dominated by any $p_2' \in (p_1, p)$ over $[\overline{p}_2 - a_2, \infty)$: for each $p_1 \geq \overline{p}_2 - a_2$, firm 2's profit is strictly increasing over $[0, p)$.

Second, I show that for $a_1 \in [\overline{a}_1', \overline{\overline{a}}_1']$ and $a_1' = \widetilde{a}_1'$, at $(a_1', a_2) = (\widetilde{a}_1', 1 - a_1)$, there is no $p_2 \in P_2^\infty$ below $p^L = \min\left\{\overline{p}_2, p_2^F(\overline{p}_2 - (a_2 - a_1'))\right\}$. Again, $\underline{p}_2 < 0$, but firm 2 can now have the incentive to undercut.

1. Every $p_2 < 0.34$ is dominated by $p_2' = 5p_2/4 < a_2 - a_1'$: for $p_1 \in [0, p_2 + (a_2 - a_1'))$, by $p_2^F(p_1) \geq p_2^F(0) > p_2'$ firm 2's profit is strictly increasing over $[0, p_2']$; for $p_1 \geq p_2 + (a_2 - a_1')$, firm 2's demand under p_2' is higher than $4/5$,[20] thus the profit is higher than p_2.

2. Every $p_1 < \overline{p}_2 - (a_2 - a_1') < 0.31$ is dominated over $[0.34, \infty)$ by $p_1' := p_1^F(p_1 + (a_2 - a_1'))$: p_1' is the unique best reply to $p_2 = p_1 + (a_2 - a_1') < \overline{p}_2$, and for $p_2 > p_1 + (a_2 - a_1')$ firm 1's profit with p_1' weakly increases, while with p_1 it remains p_1; for $p_2 \in [0.34, p_1 + (a_2 - a_1'))$, $p_1' = a_2 + p_1/2 < p_2 + (a_2 - a_1')$ is closer than p_1 to $p_1^F(p_2) = (a_1' + a_2 + p_2)/2 > a_2/2 + 3p_1/4$.

Now, since no $p_1 > p_1^*$ is rationalizable, for each $p_1' > p_1^*$ there is $m > 0$ such that there is no $p_1 \in P_1^{m-1}$ above p_1'. Let $p_1' = 0.93 > p_1^*$; I show that each $p_2 < p^L$ is dominated over $[\overline{p}_2 - (a_2 - a_1'), p_1']$, so that $p_2 \notin P_2^m$. Each $p_2 \in \left(p_1' - (a_2 - a_1'), p^L\right)$ is dominated by any $p_2' \in (p_1, p^L)$, because for each $p_1 \in [\overline{p}_2 - (a_2 - a_1'), p_1']$ firm 2's profit is strictly increasing over $\left[p_2, p^L\right)$. Each $p_2 \in [0, p_1' - (a_2 - a_1')]$ is dominated by $p_2' := 4p_2/3$: for each $p_1 \in [\overline{p}_2 - (a_2 - a_1'), p_2 + (a_2 - a_1'))$, firm 2's profit is strictly increasing over $[p_2, p_2')$ by $p_2' < 0.67 < \min\left\{p_1 + (a_2 - a_1'), p_2^F(p_1)\right\}$;[21] for each $p_1 \in [p_2 + (a_2 - a_1'), p_1']$, firm 2's demand under p_2' is higher than $3/4$.[22].

[20] The difference between p_2' and p_2 is at most 0.085, so the loss in demand is at most $a_1' + 0.0425 < 0.2$.

[21] The first inequality comes from $p_2 \leq p_1' - (a_2 - a_1') < 0.93 - 0.43$ (we have $\overline{\overline{a}}_1' < 0.42$ and $a_1' < 0.15$). The second inequality comes from $0.67 < \overline{p}_2 \leq p_1 + (a_2 - a_1')$ ($\overline{p}_2$ is minimal for $a_1 = \overline{\overline{a}}_1'$) and $p_2^F(\overline{p}_2 - (a_2 - a_1')) = 1 - (a_1' + a_2)/2 + (\overline{p}_2 - (a_2 - a_1'))/2 > 1 - 0.4 + 0.12$ ($\overline{p}_2 - (1 - a_1) + a_1'$ is minimal for $a_1 = \overline{\overline{a}}_1'$).

[22] The difference between p_2' and p_2 is less than $0.67/4 < 0.17$, so the loss in demand is less than $a_1' + 0.09 < 0.25$.

Appendix

Appendix A

A.1 Best response sets of prices

Existence at each (a_1, a_2) I am going to show that at every locations pair (a_1, a_2) there is a best response set of prices $P_1 \times P_2$, with certain characteristics.

If $a_1 = a_2$, let $P_1 \times P_2 = \{(0,0)\}$. If (a_1, a_2) is such that (p_1^*, p_2^*) is an equilibrium, let $P_1 \times P_2 = \{(p_1^*, p_2^*)\}$. Otherwise, suppose without loss of generality that $a_1 < a_2$ and $a_1 + a_2 \geq 1$; for $a_1 > a_2$ or $a_1 + a_2 < 1$, best response sets can be obtained by symmetry (see the next section for examples). The absence of pure equilibrium, $a_1 < a_2$, and $a_1 + a_2 \geq 1$ imply $a_1 > 1/4$.

For each $i = 1, 2$, let $\overline{p}_i'$ be the solution of

$$p_i - (a_2 - a_1) = x_i \cdot (p_i + (a_2 - a_1)), \tag{16}$$

where $x_1 = (1 - a_2)$ and $x_2 = a_1$. If $\overline{p}_i = \underline{p}_i$, $\overline{p}_i$ solves (16), so $\overline{p}_i = \overline{p}_i'$; else, $\overline{p}_i' < \overline{p}_i$. By $a_1 \geq 1 - a_2$, we have $\overline{p}_2' \geq \overline{p}_1'$, $\overline{p}_2 \geq \overline{p}_1$, $\underline{p}_2 \geq \underline{p}_1$.

We have $\overline{p}_1 > \underline{p}_1$ if and only if $a_1 < a_2^2$. So, let

$$\begin{aligned}
\widehat{p}_1 &= \begin{cases} \overline{p}_1' & \text{if } a_1 \geq a_2^2 \\ \overline{p}_1 & \text{if } a_1 < a_2^2 \end{cases} \\
p_2^H &= \begin{cases} \widehat{p}_1 + (a_2 - a_1) & \text{if } a_1 \geq a_2^2 \\ p_2^F(\widehat{p}_1) & \text{if } a_1 < a_2^2 \end{cases} \\
p_2^L &= \widehat{p}_1 - (a_2 - a_1) \\
p_1^L &= \begin{cases} p_1^F(p_2^L) & \text{if } p_2^H \leq \overline{p}_2 \\ \overline{p}_2 - (a_2 - a_1) & \text{if } p_2^H > \overline{p}_2 \end{cases} .
\end{aligned}$$

Given a_2, let $\overline{a}_1$ be the value of a_1 such that $p_2^L = \underline{p}_2$. So, $p_2^L < \underline{p}_2$ if $a_1 > \overline{a}_1$ and $p_2^L > \underline{p}_2$ if $a_1 < \overline{a}_1$. We have $\overline{a}_1 < a_2^2$ because, for $a_1 \geq a_2^2$, $p_2^L < \overline{p}_1' \leq \overline{p}_1 \leq \underline{p}_1 \leq \underline{p}_2$.

I am going to show that there exists a best response set $P_1 \times P_2$ such that, for some $\varepsilon, \lambda > 0$,

$$P_1 \supset [\widehat{p}_1, \widehat{p}_1 + \varepsilon), \tag{17}$$
$$P_2 \supset \left(p_2^L, p_2^L + \varepsilon\right) \cup \left[p_2^H, p_2^H + \lambda\right), \tag{18}$$

and if $a_1 < \overline{a}_1$

$$P_1 \supset \left(p_1^L, p_1^L + \varepsilon\right), \tag{19}$$

and if furthermore $p_2^H > \overline{p}_2$

$$P_2 \supset [\overline{p}_2, \overline{p}_2 + \varepsilon).$$

First I show that, for any $\varepsilon > 0$, each $p_2 \in \left(p_2^L, p_2^L + \varepsilon\right)$ is the (unique) best reply to the uniform distribution ν over $[p_1, p_1 + \delta] \subset (\widehat{p}_1, \widehat{p}_1 + \varepsilon)$ with $p_1 = p_2 + (a_2 - a_1)$ and any $\delta \in (0, p_2^L + \varepsilon - p_2)$ such that

$$p_2 > \left(1 - \frac{\gamma}{\delta} a_1\right)(p_2 + \gamma), \quad \forall \gamma \in (0, \delta], \tag{20}$$

and

$$p_2 > \frac{1}{2}\left(p_2^F(p_1 + \delta)\right)^2 \qquad \text{if } p_1 > \underline{p}_1, \tag{21}$$
$$p_2 > (1 - a_2)(p_1 + \delta + (a_2 - a_1)) \qquad \text{else.} \tag{22}$$

A small enough δ clearly satisfies (20), and also (21) and (22) because they are satisfied for $\delta = 0$, by $p_1 > \widehat{p}_1$. The right-hand side of (21) or (22) is an upper bound for firm 2's profit with $p_2' \geq p_2 + \delta$ against $p_1 + \delta$, thus also under ν. The right-hand side of (20) is an upper bound with each $p_2' \in (p_2, p_2 + \delta]$. The left-hand side is the expected profit with p_2. Hence, p_2 is the (unique) best reply to ν.

Now I show that, for some $\lambda > 0$, each $p_2 \in \left[p_2^H, p_2^H + \lambda\right)$ is the (unique) best reply to a conjecture over $[\widehat{p}_1, \widehat{p}_1 + \varepsilon)$.

Suppose first that $a_1 < a_2^2$, thus $\widehat{p}_1 = \overline{p}_1 > \underline{p}_1$. Then, p_2^H is the unique best reply to $\overline{p}_1$. Fix any $\eta \in (0,1)$. Fix $\delta \in (0,\varepsilon)$ such that

$$\widehat{p}_1 - (a_2 - a_1) \;>\; (1-\eta a_1)\left(\widehat{p}_1 + \delta - (a_2 - a_1)\right), \tag{23}$$

$$p_2^F\left(\widehat{p}_1 + \delta\right) \;<\; \widehat{p}_1 + (a_2 - a_1). \tag{24}$$

A small enough δ clearly satisfies (23), and also (24) because $\widehat{p}_1 > \underline{p}_1$. I show that each $p_2 \in (p_2^H, p_2^F(\eta\widehat{p}_1 + (1-\eta)(\widehat{p}_1+\delta)))$ is the (unique) best reply to the probability measure ν over $\{\widehat{p}_1, \widehat{p}_1+\delta\} \subset [\widehat{p}_1, \widehat{p}_1+\varepsilon)$ with mean $(p_2^F)^{-1}(p_2)$. Note that $(p_2^F)^{-1}(p_2) = \alpha\widehat{p}_1 + (1-\alpha)\left(\widehat{p}_1+\delta\right)$ for some $\alpha \in (\eta, 1)$, and let $\nu(\widehat{p}_1) = \alpha$ and $\nu(\widehat{p}_1+\delta) = 1-\alpha$. Since $\eta < \alpha$, the right-hand side of (23) is an upper bound of firm 2's expected profit under ν with $p_2' \in (\widehat{p}_1 - (a_2-a_1), \widehat{p}_1 + \delta - (a_2-a_1)]$, hence $\widehat{p}_1 - (a_2-a_1)$ is an upper bound with $p_2' \le \widehat{p}_1 + \delta - (a_2-a_1)$. Moreover, by $\widehat{p}_1 = \overline{p}_1$ we have

$$\widehat{p}_1 - (a_2 - a_1) = (p_2^F(\widehat{p}_1))^2/2 < p_2^2/2.$$

Finally, by (24), $p_2 < \widehat{p}_1 + (a_2-a_1)$, and with any $p_2' \in (\widehat{p}_1 + \delta - (a_2-a_1), \widehat{p}_1 + (a_2-a_1))$, by linearity of $\pi_2(p_2', p_1)$ in p_1, firm 2's expected profit under ν is $\pi_2(p_2', (p_2^F)^{-1}(p_2))$. Thus, p_2 is the (unique) best reply to ν.

Suppose now that $a_1^2 \ge a_2$, thus $\widehat{p}_1 = \overline{p}_1'$. I show that, for some $\lambda \in (0, \varepsilon/2)$, each $p_2 \in \left[p_2^H, p_2^H + \lambda\right)$ is the (unique) best reply to the uniform distribution over $[p_1, p_1+\delta] \subset [\widehat{p}_1, \widehat{p}_1+\varepsilon)$ with $p_1 = p_2 - (a_2 - a_1)$ and any $\delta \in (0, \varepsilon/2)$ such that,

$$p_2\left(1 - a_2 + \frac{\delta}{4}\right) \;>\; \left(1 - \frac{\gamma}{\delta}\right)\left(1 - a_2 + \frac{\delta-\gamma}{4}\right)(p_2+\gamma) \;\; \forall \gamma \in (0,\delta], \tag{25}$$

$$p_2\left(1 - a_2 + \frac{\delta}{4}\right) \;>\; \left(1 - \frac{\gamma}{\delta}a_1\right)(p_1 + \gamma - (a_2 - a_1)) \qquad \forall \gamma \in [0,\delta], \tag{26}$$

$$p_2^F\left(p_1 + \delta/2\right) \;>\; p_2. \tag{27}$$

With $p_2 = p_2^H$, thus $p_1 = \widehat{p}_1$, a small enough δ clearly satisfies (25), and also (26) and (27): by $\widehat{p}_1 = \overline{p}_1'$, $(1-a_2)\left(\widehat{p}_1 + (a_2 - a_1)\right) = \widehat{p}_1 - (a_2 - a_1)$, and by $\widehat{p}_1 \le \overline{p}_1 \le \underline{p}_1$, $p_2^F\left(\widehat{p}_1\right) \ge \widehat{p}_1 + (a_2 - a_1)$. Then, for a sufficiently small λ, they all remain satisfied. The right-hand side of (25) is firm 2's expected profit under ν with each $p_2' \in (p_2, p_2+\delta]$; the right-hand side of (26) is an upper bound with each $p_2' \in [p_1-(a_2-a_1), p_1+\delta-(a_2-a_1))$; the left-hand side is the expected profit with p_2. By (27), firm 2's expected profit is increasing over $(p_1+\delta-(a_2-a_1), p_2]$. Hence, p_2 is the (unique) best reply to ν.

If $a_1 \geq \overline{a}_1$, it only remains to show that for some $\varepsilon > 0$, each $p_1 \in [\widehat{p}_1, \widehat{p}_1+\varepsilon)$ is the (unique) best reply to some conjecture over $(p_2^L, p_2^L+\varepsilon) \cup [p_2^H, p_2^H+\varepsilon)$.

Suppose first that $a_1 > \overline{a}_1$, thus $p_2^L < \underline{p}_2$. Since firm 2 has no strict incentive to undercut $\widehat{p}_1$, firm 1, which is closer to the center, has no incentive to undercut any $p_2 < \widehat{p}_1$. Fix $\varepsilon \in (0, a_2-a_1)$ such that $p_2^L+\varepsilon < \underline{p}_2$. I show that each $p_1 \in [\widehat{p}_1, \widehat{p}_1+\varepsilon)$ is the (unique) best reply to the uniform distribution over $(p_2, p_2+\delta) \subset (p_2^L, p_2^L+\varepsilon)$[23] with $p_2 = p_1-(a_2-a_1)$ and $\delta \in (0, p_2^L+\varepsilon-p_2)$ such that

$$p_1\left(a_1+\frac{1}{4}\delta\right) \quad > \quad \left(1-\frac{\gamma}{\delta}\right)\left(a_1+\frac{1}{4}(\delta-\gamma)\right)(p_1+\gamma) \quad \forall\gamma \in (0,\delta], \tag{28}$$

$$p_1\left(a_1+\frac{1}{4}\delta\right) \quad > \quad p_2+\delta-(a_2-a_1), \tag{29}$$

A small enough δ clearly satisfies (28), and also (29), by $p_2 < \underline{p}_2$ and the strict incentive of firm 1 not to undercut. The right-hand side of (28) is firm 1's expected profit under ν with each $p_1' \in (p_1, p_1+\delta]$; the right-hand side of (29) is an upper bound with $p_1' \leq p_2+\delta-(a_2-a_1)$; the left-hand side is the expected profit with p_1. Firm 1's demand is 0 with $p_1' > p_1+\delta$. By $p_2 < \underline{p}_2$, firm 1's expected profit is increasing over $(p_2+\delta-(a_2-a_1), p_1]$. Thus, p_1 is the (unique) best reply to ν.

Suppose now that $a_1 = \overline{a}_1$. As in the $a_1 > \overline{a}_1$ case, $\widehat{p}_1$ is the (unique) best reply to a uniform distribution over $(p_2^L, p_2^L+\delta)$ for sufficiently small δ. I show that, for sufficiently small $\varepsilon > 0$, each $p_1 \in (\widehat{p}_1, \widehat{p}_1+\varepsilon)$ is the (unique) best reply to the probability measure ν over $\left\{p_2, p_2^H\right\}$ with mean $(p_1^F)^{-1}(p_1)$ for any $p_2 \in (p_1-(a_2-a_1), \min\left\{(p_1^F)^{-1}(p_1), p_2^L+\varepsilon\right\}) \neq \emptyset$. The non-emptiness comes

[23] Usually uniform distributions are defined over compact intervals. The uniform over $[p_2, p_2+\delta]$ is a valid conjecture because it gives probability 1 to $(p_2, p_2+\delta)$.

from $p_1^F(p_1-(a_2-a_1))<p_1$: by $a_1=\overline{a}_1$, $\underline{p}_2=p_2^L<p_1-(a_2-a_1)$. Moreover, $p_2^H<p_1+(a_2-a_1)$. So, p_1 is the (unique) best reply to ν if no price below $p_2^H-(a_2-a_1)$ does better. By $p_2^L<\overline{p}_1\leq\overline{p}_2$, $(p_1^F(p_2^L))^2/2>p_2^L-(a_2-a_1)$, and for sufficiently small ε, the inequality is preserved with $(p_1^F)^{-1}(p_1)$ in place of p_2^L. Since $p_2^H-(a_2-a_1)<\widehat{p}_1<p_2$, the expected profit under ν is increasing over $(p_2^L-(a_2-a_1),p_2^H-(a_2-a_1))$. Moreover, for sufficiently small ε, ν assigns sufficiently small probability to p_2^H, so that the drop in expected profit at $p_2^H-(a_2-a_1)$ (which is bounded away from $\widehat{p}_1$) does not alter the optimality of p_1.

From now on, suppose that $a_1<\overline{a}_1$. Thus $p_2^L>\underline{p}_2$, and by $p_1^L\geq p_2^L$ and $\underline{p}_2\geq\underline{p}_1$, $p_1^L>\underline{p}_1$ as well. Then, from now on, we are only going to consider prices $p_i>\underline{p}_i$. Recall that $a_1<\overline{a}_1$ implies $a_1<a_2^2$, so $\widehat{p}_1=\overline{p}_1$. There are two cases to consider: $p_2^H\leq\overline{p}_2$ and $p_2^H>\overline{p}_2$.

Let us start from $p_2^H\leq\overline{p}_2$ and recall that in this case $p_1^L=p_1^F(p_2^L)$. Let

$$\begin{aligned} p_1^H &:= p_1^F(p_2^H), \\ p_2' &:= p_1^H-(a_2-a_1)>p_2^L, \\ p_2'' &:= p_2^F(p_1^L)<p_2^H, \end{aligned}$$

We have

$$p_2'=p_1^F(p_2^H)-(a_2-a_1)>p_1^F(p_2'')-(a_2-a_1). \tag{30}$$

I show that $\left(p_1^L,p_1^H\right]\times\left(\left(p_2^L,p_2'\right)\cup\left(p_2'',p_2^H\right]\right)$ is a best response set.[24] I have already proven that:

i) each $p_2\in\left(p_2^L,p_2'\right)$ is a best reply to a conjecture over $\left(\overline{p}_1,p_1^H\right)$;

Moreover, note that:

ii) each $p_2\in\left(p_2'',p_2^H\right]$ is the best reply to $(p_2^F)^{-1}(p_2)\in\left(p_1^L,\overline{p}_1\right]$;

iii) each $p_1\in\left(p_1^L,p_1^F(p_2')\right)\cup\left(p_1^F(p_2''),p_1^H\right]$ is the best reply to $(p_1^F)^{-1}(p_1)\in\left(p_2^L,p_2'\right)\cup\left(p_2'',p_2^H\right]$ (by $p_2^H\leq\overline{p}_2$).

[24] It can be shown that $p_2'<p_2''$, otherwise, points (i)-(iii) would suffice.

It remains to show that each $p_1 \in \left[p_1^F(p_2'), p_1^F(p_2'')\right]$ is a best reply to a conjecture over $\left(p_2^L, p_2'\right) \cup \left(p_2'', p_2^H\right]$. First I show that any price $p_1 < p_2'' - (a_2 - a_1)$ is dominated over $\{p_2', p_2''\}$. We have

$$\begin{aligned} p_2' &= 1 - \frac{1}{2}a_2 + \frac{3}{2}a_1 - \sqrt{a_1}, \\ p_2'' &= \frac{1}{4}a_1 - \sqrt{a_1} + \frac{3}{2} - \frac{1}{4}a_2. \end{aligned}$$

Since $p_2' < \overline{p}_2$, any $p_1 \leq p_2' - (a_2 - a_1)$ gives to firm 1 a lower payoff than $p_1^F(p_2')$ against p_2', a fortiori against p_2''. Since $p_2'' \leq \overline{p}_2$, any p_1 with $p_2' - (a_2 - a_1) < p_1 \leq p_2'' - (a_2 - a_1)$ gives to firm 1 a lower payoff than $p_1^F(p_2'')$ against p_2''. I show that the same holds against p_2'. By (30), the profit from $p_1^F(p_2'')$ is $\pi_1(p_1^F(p_2''), p_2')$. By $a_1 + a_2 \geq 1$ and $p_2' < 1$, $p_1^F(p_2') > p_2' > p_2'' - (a_2 - a_1) \geq p_1$; thus, the profit from p_1 is bounded above by $\pi_1(p_2'' - (a_2 - a_1), p_2')$, because $\pi_1(\cdot, p_2')$ is a parabola with maximum at $p_1^F(p_2')$. But then, we have $\pi_1(p_1^F(p_2''), p_2') > \pi_1(p_2'' - (a_2 - a_1), p_2')$ if and only if

$$\begin{aligned} p_1^F(p_2'') - p_1^F(p_2') &< p_1^F(p_2') - p_2'' + (a_2 - a_1) \Leftrightarrow \\ \frac{1}{4} - \frac{5}{8}a_1 + \frac{1}{8}a_2 &< \frac{3}{2}a_2 - 1 + \frac{1}{2}\sqrt{a_1} \Leftrightarrow \\ \frac{11}{8}a_2 + \frac{5}{8}a_1 + \frac{1}{2}\sqrt{a_1} &> \frac{5}{4} \end{aligned}$$

and the last inequality is true by $a_1 + a_2 \geq 1$ and $a_1 > 1/4$.

Now, fix $p_1 \in \left[p_1^F(p_2'), p_1^F(p_2'')\right]$. Inequality (30) and the dominance relations are preserved for some $\delta > 0$ and $p_2' - \delta > p_2^L$ and $p_2'' + \delta < p_2^H$ in place of p_2' and p_2''. Therefore, by linearity of $\pi_1(\cdot, p_2)$ in p_2, p_1 is the best reply to the conjecture over $\{p_2' - \delta, p_2'' + \delta\}$ with mean $(p_1^F)^{-1}(p_1) \in [p_2', p_2'']$.

Finally, suppose that $p_2^H > \overline{p}_2$. Recall that in this case, $p_1^L = \overline{p}_2 - (a_2 - a_1)$.

Let

$$\begin{aligned}
p_1^H &:= p_1^F(\overline{p}_2) \\
p_2' &:= p_1^H - (a_2 - a_1) < \overline{p}_2 \\
p_2'' &:= p_2^F(p_1^L) < \overline{p}_2 \\
p_1' &:= p_2^H - (a_2 - a_1) < \overline{p}_1 \\
p_1'' &:= p_1^F(p_2^L) < \overline{p}_1
\end{aligned}$$

I show that for each $i = 1, 2$, $\times_{i=1,2}\left(\left(p_i^L, p_i'\right) \cup \left(p_i'', p_i^H\right]\right)$ is a best response set.[25] I have already proven that:

i) each $p_i \in \left(p_i^L, p_i'\right)$ is a best reply to a conjecture over $\left(\overline{p}_{-i}, p_{-i}^H\right)$.

Moreover, note that

ii) each $p_i \in \left(p_i'', p_i^F(p_{-i}')\right) \cup \left(p_i^F(p_{-i}''), p_i^H\right]$ is the best reply to $(p_i^F)^{-1}(p_i) \in \left(p_{-i}^L, p_{-i}')\right) \cup \left(p_{-i}'', \overline{p}_{-i}\right]$.

It remains to show that each $p_i \in \left[p_i^F(p_{-i}'), p_i^F(p_{-i}'')\right]$ is a best reply to a conjecture over $\left\{p_{-i}' - \delta, p_{-i}'' + \delta\right\}$ for some $\delta > 0$. I am going to show that each $p_i \leq p_{-i}'' - (a_2 - a_1)$ is dominated over $\left\{p_{-i}', p_{-i}''\right\}$ by $p_i^F(p_{-i}'')$. Then, the proof follows the same lines as for the $p_2^H < \overline{p}_2$ case.

The inequality $p_2^H > \overline{p}_2$ reads

$$2 - 2\sqrt{a_1} > 4 - a_2 - a_1 - 4\sqrt{1 - a_2},$$

which, given $a_1 > 1/4$, requires $a_2 < 3/4$. By $a_1 < \overline{a}_1$, we have $p_2^L > \underline{p}_2$, which reads

$$2 + 2a_1 - 4\sqrt{a_1} > 3a_1 - a_2,$$

and yields

$$\sqrt{a_1} < \frac{-4 + \sqrt{16 + 8 + 4a_2}}{2},$$

[25] It can be shown that $p_2' < p_2''$ and $p_1' < p_1''$, otherwise, points (i)-(ii) would suffice.

which given $a_2 < 3/4$ implies $a_1 < 0.36$. By $a_1 < a_2^2$ and $a_1 > 1/4$, we have $a_2 > 1/2$. To recap, we have $a_1 + a_2 \geq 1$, $a_1 \in (1/4, 0.36)$, and $a_2 \in (1/2, 3/4)$. This also implies $\overline{p}_1 \leq \overline{p}_2 < 1$.

By $p''_{-i} < \overline{p}_{-i}$, $p_i^F(p''_{-i}) < p_i^F(\overline{p}_{-i}) = p_i^H = p'_{-i} + (a_2 - a_1)$. So, as before, it remains to show that $p_i^F(p''_{-i}) - p_i^F(p'_{-i}) < p_i^F(p'_{-i}) - p''_{-i} + (a_2 - a_1)$ for $i = 1, 2$. For $i = 2$, we have

$$\begin{aligned} x &:= p_2^F(p'_1) = \frac{3}{2} + \frac{1}{4}\overline{p}_1 - \frac{1}{4}a_1 - \frac{5}{4}a_2, \\ y &:= p_2^F(p''_1) = 1 - \frac{1}{2}a_2 + \frac{1}{4}\overline{p}_1, \\ z &:= p''_1 - (a_2 - a_1) = \frac{1}{2}\overline{p}_1 - a_2 + 2a_1, \\ y + z - 2x &= \frac{1}{4}\overline{p}_1 + a_2 + \frac{5}{2}a_1 - 2 < 0, \end{aligned}$$

where the desired inequality comes from $a_2 < 3/4$, $a_1 < 0.36$ and $\overline{p}_1 < 1$. For $i = 1$, we have

$$\begin{aligned} x &:= p_1^F(p'_2) = \frac{1}{4}\overline{p}_2 + \frac{5}{4}a_1 + \frac{1}{4}a_2 \\ y &:= p_1^F(p''_2) = \frac{1}{2} + \frac{1}{2}a_1 + \frac{1}{4}\overline{p}_2 \\ z &:= p''_2 - (a_2 - a_1) = 1 + \frac{1}{2}\overline{p}_2 - 2a_2 + a_1 \\ y + z - 2x &= \frac{3}{2} + \frac{1}{4}\overline{p}_2 - a_1 - \frac{5}{2}a_2 < 0, \end{aligned}$$

where the desired inequality comes from $a_1 + a_2 \geq 1$, $a_2 > 1/2$, and $\overline{p}_2 < 1$.

A.2 Extensive-form rationalizability

Let $A' = \{0.31, 0.32\}$ and $A'' = \{0.68, 0.69\}$. Fix $a_2 \in A''$; using the results of Appendix A.1, I am going to fix a best response set for any value of a_1. (Note that by $a_2 \leq 0.69$, (p_1^*, p_2^*) is never an equilibrium.)

If $a_1 > a_2$, fix a best response set $\overline{P}_1 \times \overline{P}_2$ with $\overline{P}_2 \supset [\widehat{p}_2, \widehat{p}_2 + \varepsilon)$ for some

$\varepsilon > 0$. By symmetry, it exists and we have

$$\widehat{p}_2 \leq \overline{p}_2 = 2 + a_1 + a_2 - 4\sqrt{a_2}. \tag{31}$$

If $a_1 = a_2$, let $\overline{P}_1 \times \overline{P}_2 = \{0, 0\}$.

If $a_1 \in (0.32, a_2)$, fix a best response set $\overline{P}_1 \times \overline{P}_2$ with $\overline{P}_2 \supset \left(p_2^L, p_2^L + \varepsilon\right) \cup \left[p_2^H, p_2^H + \varepsilon\right)$ for some $\varepsilon > 0$, and recall that

$$\begin{aligned}
p_2^H &= \overline{p}_1' + (a_2 - a_1) = \frac{(2 - a_2)(a_2 - a_1)}{a_2} + (a_2 - a_1) = 2 - 2\frac{a_1}{a_2}, && (32)\\
p_2^L &= \overline{p}_1' - (a_2 - a_1) = p_2^H - 2(a_2 - a_1) \qquad \text{if } a_1 \in (a_2^2, a_2);\\
p_2^H &= p_2^F(\overline{p}_1) = 2 - 2\sqrt{a_1}, && (33)\\
p_2^L &= \overline{p}_1 - (a_2 - a_1) = 2 + 2a_1 - 4\sqrt{a_1} \qquad \text{if } a_1 \in (0.32, a_2^2].
\end{aligned}$$

If $a_1 \in (0, 0.32]$, fix a best response set $\overline{P}_1 \times \overline{P}_2$ with $\overline{P}_1 \supset \left(p_1^L, p_1^L + \varepsilon\right)$ and $\overline{P}_2 \supset \left(p_2^L, p_2^L + \varepsilon\right) \cup [\overline{p}_2, \overline{p}_2 + \varepsilon)$ for some $\varepsilon > 0$, where, letting $\overline{a}_1' < 0.30$ be the value of a_1 such that $p_1^F(\overline{p}_2) = \overline{p}_1$,[26]

$$\begin{aligned}
p_1^L &= \overline{p}_2 - (a_2 - a_1), && (34)\\
p_2^L &= \overline{p}_1 - (a_2 - a_1) = 2 + 2a_1 - 4\sqrt{a_1} \qquad \text{if } a_1 \in (\overline{a}_1', 0.32], && (35)\\
p_2^L &= p_2^F(p_1^L) = 3 - \frac{1}{2}a_1 - \frac{3}{2}a_2 - 2\sqrt{1 - a_2} \qquad \text{if } a_1 \in [0, \overline{a}_1']. && (36)
\end{aligned}$$

To see that that $\overline{P}_1 \times \overline{P}_2$ exists, consider first $a_1 \in [1 - a_2, 0.32]$. We have

$$p_2^H = p_2^F(\overline{p}_1) = 2 - 2\sqrt{a_1} > 4 - a_2 - a_1 - 4\sqrt{1 - a_2} = \overline{p}_2.$$

Moreover, $\overline{a}_1 > 0.32$, because for $a_1 = 0.32$, $p_2^L > \underline{p}_2$. Then, existence follows verbatim from the results of Appendix A.1. Consider now $a_1 \in [0, 1 - a_2)$. We have $\overline{p}_2 > \underline{p}_2$, thus $\widehat{p}_2 = \overline{p}_2$, and $\overline{p}_2 - (a_2 - a_1) > \underline{p}_1$.[27] Then, the existence of $\overline{P}_1 \times \overline{P}_2$ follows from the results of Appendix A.1 with the roles of the two

[26] For $a_1 = 0.30$, we have $p_1^F(\overline{p}_2) = 2 - 2\sqrt{1 - a_2} > 2.30 + a_2 - 4\sqrt{0.30} = \overline{p}_1$.

[27] The first condition is verified by $1 - a_2 < (1 - a_1)^2$. The second is verified at $(1 - a_2, a_2)$ (see Section 3.2.2) and the left-hand side is independent of a_1, while the right-hand side decreases as a_1 decreases.

firms inverted (case $p_1^H = p_1^F(\overline{p}_2) > \overline{p}_1$ for $a_1 > \overline{a}_1'$, case $p_1^H = p_1^F(\overline{p}_2) \leq \overline{p}_1$ else).

For $a_1 \in [1-a_2, a_2)$, we have $p_2^L < \widehat{p}_1 \leq \overline{p}_1 \leq \overline{p}_2$. For $a_1 \in (\overline{a}_1', 1-a_2)$, we have $p_2^L = \overline{p}_1 - (a_2 - a_1) < \overline{p}_2$. For $a_1 \in [0, \overline{a}_1']$, we have $p_2^L = p_2^F(\overline{p}_2 - (a_1 - a_1)) < \overline{p}_2$. We conclude that $p_2^L < \overline{p}_2$ for all $a_1 \in [0, a_2)$.

Now, for each $a_1 \leq 1/2$, I fix a "optimistic" distribution over $\overline{P}_2$ and a best reply of firm 1.

Fix $a_1 \in [0, 0.32]$. At (a_1, a_2), fix a uniform distribution ν^{a_1} over $[\overline{p}_2 + \delta, \overline{p}_2 + \delta + \gamma] \subset \overline{P}_2$ for sufficiently small $\delta, \gamma > 0$ so that, letting $p_1^O := \overline{p}_2 + \delta - (a_2 - a_1)$, (i) $p_1^O \in \overline{P}_1$, (ii) $p_1^O < \overline{p}_1$, and (iii) p_1^O best replies to ν^{a_1} (see Appendix A.1 for the construction of ν^{a_1}). Note that (ii) can be satisfied because $\overline{p}_2 - \overline{p}_1 < a_2 - a_1$. By (34), $p_1^O = p_1^L + \delta$.

Fix $a_1 \in (0.32, 1/2]$. At (a_1, a_2), fix a uniform distribution ν^{a_1} over $\left[p_2^H, p_2^H + \gamma\right] \subset \overline{P}_2$ for sufficiently small $\gamma > 0$ so that firm 1 has a best reply p_1^O.[28] Firm 1's expected profit under ν^{a_1} is bounded below by

$$a_1 \cdot \left(p_2^H(a_1, a_2) + (a_2 - a_1)\right) > \frac{1}{2}\left(\frac{a_1' + a_2}{2} + \frac{p_2^L(a_1', a_2)}{2}\right)^2 \quad \forall a_1' \in [0, a_2)\,, \tag{37}$$

where the right-hand side is an upper bound of firm 1's profit against $p_2^L < \overline{p}_2$ at (a_1', a_2). The inequality holds because it holds for $a_1 = 1/2$ and $a_1' = \overline{a}_1' < 0.30$, which, respectively, minimize and maximize the two sides. By (32) and (33),

$$a_1 \cdot \left(p_2^H(a_1, a_2) + (a_2 - a_1)\right) = \begin{cases} 2a_1 - 2a_1\sqrt{a_1} + a_1 a_2 - a_1^2 & \text{if } a_1 \in (0.32, a_2^2] \\ 2a_1 - 2\frac{a_1^2}{a_2} + a_1 a_2 - a_1^2 & \text{if } a_1 \in (a_2^2, 1/2] \end{cases},$$

and note that $-2a_1\sqrt{a_1} \geq -2a_1^2/a_2$ for $a_1 \leq a_2^2$, and $2a_1 - 2a_1^2/a_2 + a_1 a_2 - a_1^2$ is concave and lower for $a_1 = 1/2$ than for $a_1 = 0.32$. The right-hand side is maximized by $a_1' = \overline{a}_1'$ because, by (35) and (36),

$$p_2^L(\overline{a}_1', a_2) + \overline{a}_1' > p_2^L(a_1', a_2) + a_1', \quad \forall a_1' \in [0, a_2)\,, a_1' \neq \overline{a}_1'.$$

[28] The best reply can be $p_2^H - (a_2 - a_1)$, or $p_1^F(p_2^H + \delta/2)$, or $p_2^H + (a_2 - a_1)$; which one is immaterial for the analysis.

Now, for any a_1', I fix a low price in $\overline{P}_2$. For each $a_1' \in [0, a_2)$, at (a_1', a_2), let $p_2^D := p_2^L + \delta$ for sufficiently small $\delta > 0$ so that (i) $p_2^D \in \overline{P}_2$, (ii) $p_2^D < \overline{p}_2$, and (iii) inequality (37) still holds with $a_1 = 1/2$ and $p_2^D = p_2^L(a_1', a_2) + \delta$ in place of $p_2^L(a_1', a_2)$. Note that (ii) can be satisfied because for $a_1' \leq \overline{a}_1'$, $p_2^L = p_2^F(p_1^L)$ and $p_1^L = \overline{p}_2 - (a_2 - a_1) > \underline{p}_1$, and for $a_1 > \overline{a}_1'$, $p_2^L = \overline{p}_1 - (a_2 - a_1)$ and $\overline{p}_1 - \overline{p}_2 < a_2 - a_1$. For each $a_1' \in (a_2, 1]$, let $p_2^D = \widehat{p}_2 \in \overline{P}_2$. Let $p_2^D(a_2, a_2) = 0$.

Now, I show that for each $a_1 \in [0, 1/2]$, firm 1's expected profit at (a_1, a_2) under the optimistic distribution ν^{a_1} is higher than against the low price p_2^D at any (a_1', a_2) with $a_1' < a_2$. (For $a_1' > a_2$, this is easy to see.) If $a_1 \in (0.32, 1/2]$, this is true by construction of each $a_1' \in [0, a_2)$; else, ν^{a_1}'s support is above $\overline{p}_2$, firm 1's profit at (a_1', a_2) against $\overline{p}_2$ is independent of a_1', and $p_2^D < \overline{p}_2$.

Then, choosing a_1 and $p_1^O(a_1, a_2)$ is optimal given some distribution over

$$S_2^{a_1,a_2} := \left\{ s_2 \in S_2(a_2) \;\middle|\; \begin{array}{l} s_2(a_1) \in P_2(a_1, a_2) \\ \forall a_1' \neq a_1, s_2(a_1') = p_2^D(a_1', a_2) \end{array} \right\},$$

where $P_1(a_1, a_2) \times P_2(a_1, a_2)$ denotes the best response set $\overline{P}_1 \times \overline{P}_2$ at (a_1, a_2).

For each $a_1 \in A'$ and $a_2 \in [1/2, 1]$, $\overline{P}_1 \times \overline{P}_2$, p_1^D, and p_2^O can be defined at (a_1, a_2) symmetrically. In particular, if $a_2 \in A''$, by $1 - a_2 \leq 0.32$, p_2^O is constructed at (a_1, a_2) as $p_2^O = \overline{p}_2^L + \delta$ for sufficiently small δ. At (a_1, a_2), p_2^D has been constructed above as $p_2^L + \delta$ for sufficiently small δ. But then, we can obtain $p_2^O = p_2^D$ by constructing them with the same δ.

For each $a_1 \in A''$ and $a_2 \in [0, 1/2]$, and for each $a_2 \in A'$ and $a_1 \in [1/2, 1]$, construct $\overline{P}_1 \times \overline{P}_2$, p_1^D, and p_2^O in the same way by symmetry.

For any other $(a_1, a_2) \in [0, 1]^2$, let $P_1(a_1, a_2) \times P_2(a_1, a_2)$ be the best response set constructed in Appendix A.1.

Now, for each $i = 1, 2$, consider the following sets of plans:

$$S_i^* = \left\{ s_i \in S_i \;\middle|\; \begin{array}{c} \left(s_i^{\emptyset} \leq 1/2\right) \Rightarrow \left(\exists a_{-i} \in A'', s_i(a_{-i}) = p_i^O(s_i^{\emptyset}, a_{-i})\right) \\ \left(s_i^{\emptyset} > 1/2\right) \Rightarrow \left(\exists a_{-i} \in A', s_i(a_{-i}) = p_i^O(s_i^{\emptyset}, a_{-i})\right) \\ \forall a_{-i}' \neq a_{-i}, s_i(a_{-i}') \in P_i(s_i^{\emptyset}, a_{-i}') \end{array} \right\}.$$

For each $a_2 \in A''$ and $a_1 \in [0, 1/2]$, observe that $S_2^{a_1,a_2} \subset S_2^*$, because for each

$s_2 \in S_2^{a_1,a_2}$, there is $a_1' \in A'$ such that $s_2(a_1') = p_2^D(a_1', a_2) = p_2^O(a_1', a_2)$.[29]

I am going to show that for each $s_i \in S_i^*$, there is $\mu_i \in \Delta^H(S_{-i}^*)$ such that $s_i \in \rho_i(\mu_i)$. Since every locations pair is induced by some $(s_1, s_2) \in S_1^* \times S_2^*$, we have $\mu_i(S_{-i}^*|a) = 1$ for each $a \in [0,1]^2$. Then, it is immediate to see by induction that $S_i^* \subset S_i^\infty$ for each $i = 1, 2$.

Without loss of generality, fix $s_1 \in S_1^*$ with $s_1^\emptyset \leq 1/2$. Fix $a_2 \in A''$ such that $s_1(a_2) = p_1^O(s_1^\emptyset, a_2)$. By construction, $s_1^\emptyset$ and $p_1^O(s_1^\emptyset, a_2)$ are optimal given a distribution ν over $S_2^{s_1^\emptyset, a_2} \subset S_2^*$. For each $a_2' \neq a_2$ and $p_2 \in P_2(s_1^\emptyset, a_2')$, there is $s_2' \in S_2(a_2')$ with $s_2'(s_1^\emptyset) = p_2$ and $s_2'(a_1) = p_2^O(a_1, a_2')$ for some $a_1 \in (A' \cup A'') \setminus \{s_1^\emptyset\}$, so that $s_2 \in S_2^*$. Then, there is $\mu_1 \in \Delta^H(S_2^*)$ with $\mu_1(\cdot|\emptyset) = \nu$ such that $\mu_1(\cdot|s_1^\emptyset, a_2')$ justifies $s_1(a_2') \in P_1(s_1^\emptyset, a_2')$ for each $a_2' \neq a_2$, so that $s_1 \in \rho_1(\mu_1)$.

To conclude, note that, for $S_1^{es} \times S_2^{es}$ defined in Section 3.1, $S_i^* \cap S_i^{es} \neq \emptyset$ for each $i = 1, 2$. This is because $p_1^O(1/4, a_2') < \overline{p}_1(1/4, a_2')$ for each $a_2' \in A''$; and for each $a_2' \notin A''$, $P_1(1/4, a_2')$ includes prices that can be prescribed by S_1^{es}.

Appendix B

I show that there is $p \in P$ with $p \leq \overline{p}$. I will prove that $p^I \leq \overline{p}$; then, if $p^I < \overline{p}$, the result follows, if $p^I = \overline{p}$, $\overline{p} \in P$ for the following reason. Suppose $\overline{p} \notin P$; thus, P contains a sequence σ of prices that converges to $\overline{p}$ from above. But then, P also contains a sequence σ' that converges to $\overline{p} + (a_2 - a_1)$ from above, and I will show that a conjecture over σ' justifies $\overline{p}$, a contradiction. To see why σ requires σ', suppose by contradiction that $P \cap (\overline{p} + (a_2 - a_1), \overline{p} + (a_2 - a_1) + \varepsilon) = \emptyset$ for some $\varepsilon > 0$. Then, for any conjecture over P, the expected profit of the

[29] The importance of this fact is the following. Firm 1 may think firm 2 locates at $a_2 \in A''$ precisely in the hope that firm 1 locates at $a_1' \in A'$, and then fixes a price that best replies to an optimistic conjecture, rather than being surprised by a_1', and then fixing any price in $P_2(a_1', a_2)$. This could make it impossible to justify $a_1 \notin A'$ with a belief over S_2^*, where all the plans are justified precisely under the belief that firm 1 locates at some $a_1' \in A'$. But thanks to $p_2^D(a_1', a_2) = p_2^O(a_1', a_2)$, firm 1 can believe that firm 2 will fix the low price $p_2^D(a_1', a_2)$ as a best reply to the optimistic conjecture, deterring a deviation from a_1 to a_1'.

respondant is strictly increasing over $(\overline{p}, \min\left\{p^F(\overline{p}), \overline{p}+\varepsilon\right\})$, where $p^F(\overline{p}) > \overline{p}$ by $\overline{p} < p^* = 1$, so the prices of σ in this interval could not be justified in P.

Now I prove that $p^I \leq \overline{p}$. Suppose that there exist $p \in P$ and $\varepsilon > 0$ such that $p > \overline{p}$ and $[p, p+\varepsilon) \subseteq P$. Then, for each $p' \in [p, p+\varepsilon)$, $p' - (a_2 - a_1)$ is the best reply to a uniform distribution over $[p', p'+\delta]$ for sufficiently small δ (see Appendix A.1 for details). But then, we have $[p-(a_2-a_1), p+\varepsilon-(a_2-a_1)) \subseteq P$. Iterating if necessary, we find prices smaller or equal to $\overline{p}$ in P. Suppose from now on that $p^I > \overline{p}$; I show that then such p and ε exist, a contradiction.

Any price $p > 4(a_2 - a_1)$ is dominated over $[0, p]$ by any $p' \in (p/2, p - 2(a_2 - a_1))$: whenever p gives positive demand, it is at most $1/2$, and $p' > p/2$ gives demand 1. Then, p is also dominated over $[0, p+\varepsilon]$ for some $\varepsilon > 0$, thus $\sup P \leq 4(a_2 - a_1)$. Moreover, $\overline{p} > 2(a_2 - a_1)$, because, by $a_1 > 1/4$, $3 - 4\sqrt{a_1} > 2 - 4a_1$ (recall $a_1 + a_2 = 1$). Hence, $P \subseteq (2(a_2 - a_1), 4(a_2 - a_1)]$.

For each conjecture over prices ν, let p^ν be the mean of ν.

Consider $p > \overline{p}$ and a sequence of prices $(p^k)_{k=0}^\infty$ that converges to p from above. Construct a probability measure ν by assigning probability $(1/2)^j$ to each $\widetilde{p}^j$ of a decreasing subsequence $(\widetilde{p}^j)_{j=1}^\infty$ of $(p^k)_{k=0}^\infty$ such that

$$p - (a_2 - a_1) \;>\; \left(\widetilde{p}^j - (a_2 - a_1)\right)\left(\textstyle\sum_{l=1}^{j} \dfrac{1}{2^l} + a_2 \sum_{l>j} \dfrac{1}{2^l}\right), \;\forall j \geq 1; \tag{38}$$

$$p - (a_2 - a_1) \;>\; \frac{1}{2}\left(\frac{1}{2} + \frac{1}{2}\widetilde{p}^1\right)^2. \tag{39}$$

Such subsequence exists: (39) is satisfied by picking $\widetilde{p}^1$ sufficiently close to p, since it holds with p in place of $\widetilde{p}^1$ by $p > \overline{p}$; (38) is satisfied by picking each subsequent $\widetilde{p}^j$ close enough to p as well. The right-hand side of inequality (38) (resp., (39)) is an overestimation of the expected profit given by any $p' \in (\widetilde{p}^{j+1} - (a_2 - a_1), \widetilde{p}^j - (a_2 - a_1)]$ (resp., by any $p' > \widetilde{p}^1 - (a_2 - a_1)$) under ν, while the left-hand side is the profit given by $p - (a_2 - a_1)$. Thus, the best reply to ν is $p - (a_2 - a_1)$. Then, there is no sequence of prices in P that converges to $p^I > \overline{p}$ from above, otherwise $p^I - (a_2 - a_1) \in P$, a contradiction. But then, $p^I \in P$, and there is $\widetilde{p} > p^I$ such that $P \cap (p^I, \widetilde{p}) = \emptyset$.

Price p^I can be a best reply to a conjecture over P only if there is a decreasing sequence of prices $(p^k)_{k=0}^{\infty}$ that converges to $p^I + (a_2 - a_1)$, otherwise the respondant's expected profit would be strictly incresasing over $\left[p^I, \min\left\{p^F(p^I), p^I + \varepsilon\right\}\right)$ for some $\varepsilon > 0$ ($p^F(p^I) > p^I$ because, since no price above p^* is rationalizable, $p^I < \sup P \leq p^* = 1$). So, fix a decreasing sequence $(p^k)_{k=0}^{\infty}$ in P that converges to $p^I + (a_2 - a_1)$ with $p^0 < \widetilde{p} + (a_2 - a_1)$. Each price p^k is a best reply to a conjecture ν over $P \backslash \left\{p^I\right\}$. To see this, fix a conjecture ν' over P that justifies p^k, and note that p^k is a worst reply to p^I (it gives 0 demand), so it must be a best reply to $\nu' | \left(P \backslash \left\{p^I\right\}\right)$. Since p^k is at less than $(a_2 - a_1)$ distance from any $p' \in P \backslash \left\{p^I\right\}$, by linearity of $\pi_i(p_i, p_{-i})$ in p_{-i}, ν must have mean $(p^F)^{-1}(p^k)$ for p^k to be best a reply.

Suppose now that there exist $p' \in P$ and $\gamma > 0$ such that $p' > \inf \mathrm{Supp} \nu$ and the expected profit under ν with p^k is higher by at least γ than with any price in a neighbourhood of $p' - (a_2 - a_1)$. Fix $p'' < p'$ such that $\nu([p'', p')) > 0$ and

$$(p' - (a_2 - a_1))\, \nu((p'', p')) < \gamma, \tag{40}$$

Construct ν' from ν by moving from (p'', p') to p' the probability $\nu((p'', p'))$ if $\nu((p'', p')) > 0$, else by moving from p'' to p' probability $\varepsilon := \min\{\nu(p''), \gamma\}$. With respect to ν, the new conjecture ν' has higher mean, the expected profit with any $p < p'' - (a_2 - a_1)$ is unchanged, with any $p \in [p'' - (a_2 - a_1), p' - (a_2 - a_1)]$ has gone up less than γ by (40), and with any $p \in (p' - (a_2 - a_1), \sup P - (a_2 - a_1))$ has gone up less than with $p^k > \sup P - (a_2 - a_1)$: same increase in expected demand but at a lower price. So, the best reply to ν' is $p^F(p^{\nu'})$. The exercise can be repeated by raising p'' or lowering ε and the desired interval $\left[p^k, p^F(p^{\nu'})\right) \subset P$ obtains.

To complete the proof, it remains to show that, for some p^k and the corresponding ν, such p' exists. Suppose not. This means that each p^k is justified by a conjecture ν^k such that, for each $p' \in P$ with $p' > \inf \mathrm{Supp} \nu^k$, the expected profit from prices below $p' - (a_2 - a_1)$ is not bounded away from the expected profit $(p^k)^2/2$ given by p^k. Note that (i) $\inf \mathrm{Supp} \nu^k \leq \left(p^k\right)^2/2 + (a_2 - a_1)$, otherwise prices slightly below $\inf \mathrm{Supp} \nu^k - (a_2 - a_1)$ would bring under ν^k

expected profit higher than $(p^k)^2/2$. On the other hand, we must have either (ii) $\inf\text{Supp}\nu^k = \inf\text{Supp}\nu^{k+1}$, or (iii) $\inf\text{Supp}\nu^k \geq \left(p^{k+1}\right)^2/2 + (a_2 - a_1)$, otherwise all prices below some $p' \in P$ with $p' > \inf\text{Supp}\nu^{k+1}$ would bring under ν^{k+1} expected profit bounded away from $\left(p^{k+1}\right)^2/2$. If for every $j \geq 0$ there is $k \geq j$ such that (iii) holds, together with (i) we obtain a decreasing sequence of prices in P converging to a point below $p^I + (a_2 - a_1)$ (each ν^k must give probability to prices below p^k to justify p^k). As observed, such sequence cannot exist, hence there is $j \geq 0$ that for each $k \geq 0$, $\inf\text{Supp}\nu^k = \inf\text{Supp}\nu^{k+1}$. So, fix $j < k < l$ such that p^j, p^k, and p^l are justified by conjectures ν^j, ν^k, ν^l with

$$\widetilde{p} \leq p^{IS} := \inf \text{Supp}\nu^j = \inf \text{Supp}\nu^k = \inf \text{Supp}\nu^l = \inf \text{Supp}\nu^{l+1} < \left(p^l\right)^2/2 + (a_2 - a_1)$$

and $P \cap \left(p^{IS}, p^{IS} + \varepsilon\right) = \emptyset$ for some $\varepsilon > 0$, otherwise some p' would not satisfy the requirement. Consider the convex combination $\nu := \alpha\nu^j + (1-\alpha)\nu^l$ with $\alpha p^{\nu^j} + (1-\alpha)p^{\nu^l} = p^{\upsilon^k} = (p^F)^{-1}(p^k)$. For each $p' \in P$ with $p' > p^{IS}$, prices slightly below $p' - (a_2 - a_1)$ bring under ν a strictly higher profit than p^k, since they bring expected profit not bounded away from the optimum under ν^j and ν^l (p' satisfies the requirement with ν^j and ν^l), while the expected profit of p^k under ν^j and ν^l is strictly below $\left(p^j\right)^2/2$ and $\left(p^l\right)^2/2$. I am going to show that then p^k is not a best reply to υ^k, a contradiction. Conjectures ν and ν^k have the same mean, p^{ν^k}, and the same infimum of the support, p^{IS}. For p^k to be a best reply to ν^k, we need that for no $p' > p^{IS}$ prices slightly below $p' - (a_2 - a_1)$ bring a strictly higher profit. But since ν^k and ν have the same mean, one cannot first-order stochastically dominate the other, so there must be $p' \in P$ with $p > p^{IS}$ and $\nu^k([p', \infty)) \geq \nu([p', \infty))$. The expected profit of a price under a conjecture with a given mean is increasing in the probability to take over the whole market. Therefore, the expected profit of some price slightly below $p' - (a_2 - a_1)$ is still strictly higher than $(p^k)^2/2$ under ν^k. So, p^k cannot be a best reply to ν^k.

3.3 Appendix C — Extensive-form best response sets

An Extensive-Form Best Response Set (Battigalli and Friedenberg, 2012; henceforth EFBRS) is a set of pairs of plans $\widehat{S} = \widehat{S}_1 \times \widehat{S}_2$ that satisfies the following condition: for each $i = 1, 2$ and $s_i \in \widehat{S}_i$, there is $\mu_i \in \Delta^H(\widehat{S}_{-i})$ such that $s_i \subseteq \rho_i(\mu_i) \subseteq \widehat{S}_i$.[30] I am going to show that for each symmetric locations pair (a_1, a_2) with $a_1 \leq \overline{\overline{a}}_1 \simeq 0.38$, there is an EFBRS $\widehat{S} = \widehat{S}_1 \times \widehat{S}_2 \subseteq S_1(a_1) \times S_2(a_2)$. An EFBRS can be constructed also for higher values of a_1, but not all the way up to 1/2; the construction is left to the reader.

Fix first $a_1 \leq 1/4$. For each $i = 1, 2$, define s_i as $s_i^\emptyset = a_i$, $s_i(a_{-i}) = 1$, and, for each $a'_{-i} \neq a_{-i}$, $s_i(a'_{-i}) = \min\left\{\left|a'_{-i} - a_i\right|, p_i^F(0)\right\}$, where $p_i^F(0)$ is meant at $\left(a_i, a'_{-i}\right)$. Let $\widehat{S}_i = \{s_i\}$. We have $\{s_i\} = \rho_i(\mu_i)$ for any $\mu_i \in \Delta^H(\widehat{S}_{-i})$ (thus $\mu_i(s_{-i}|\emptyset) = 1$) such that for each $a'_{-i} \neq a_{-i}$, the price distribution induced by $\mu_i(\cdot|a_i, a'_{-i})$ justifies only $s_i(a'_{-i})$:[31] against s_{-i}, firm 1's profit at (a_1, a_2) is 1/2, while at each (a'_i, a_{-i}) with $a'_i \neq a_i$ it is lower than 1/2. This is because, if $a'_1 < a_2 < 1$, an upper bound of firm 1's profit against $p_2 = a_2 - a'_1$ is[32]

$$\pi_1(p_1^F(p_2), p_2) = \frac{1}{2}\left(\frac{a'_1 + a_2}{2} + \frac{a_2 - a'_1}{2}\right)^2 = \frac{1}{2}a_2^2 < \frac{1}{2}, \tag{41}$$

and if $a_2 = 1$, $s_2(a'_1) = p_2^F(0) < a_2 - a'_1$; if $a'_1 > a_2$ and for firm 2, symmetric arguments apply.

Now, fix $a_1 \in (0.25, \overline{a}_1]$, where $\overline{a}_1 \simeq 0.34$ has been defined in Section 3.2.2 as the value of a'_1 such that, at $(a'_1, 1 - a'_1)$, $\overline{p}_2 - (1 - 2a'_1) = \underline{p}_2$. For $a_1 < a_2^2$, in Appendix A.1 I construct a best response set of prices $P_1 \times P_2$ where $\inf P_1 > \inf P_2 = \overline{p}_1 - (a_2 - a_1) \notin P_2$, and where each price is the unique best reply to some conjecture.[33] By symmetry ($a_2 = 1 - a_1$), we also have a

[30] This is a strengthening of Self-Justifiability, however the two conditions are generically equivalent.

[31] If $s_i(a'_{-i}) = \left|a'_{-i} - a_i\right|$, it is the only best reply to a uniform distribution over $[0, \varepsilon]$ for sufficiently small ε (see Appendix A.1 for the construction of analogous uniform distributions).

[32] Obviously, firm 1 cannot undercut $p_2 = a_2 - a'_1$, as the price would be negative.

[33] This can be seen by inspection of the proofs.

symmetric best response set of prices $P'_1 \times P'_2$. For each $i = 1, 2$, let

$$\widehat{S}_i := \left\{ s_i \in S_i(a_i) \;\middle|\; \begin{array}{c} \exists p_i \in P_i \cup P'_i, s_i(a_{-i}) = p_i \\ \forall a'_{-i} \neq a_{-i}, s_i(a'_{-i}) = \min\left\{ \left| a'_{-i} - a_i \right|, p_i^F(0) \right\} \end{array} \right\}. \quad (42)$$

By (41), at (a'_1, a_2) with $a'_1 < a_2$, an upper bound of firm 1's profit against $p_2 = a_2 - a'_1$ is $a_2^2/2$; for $a'_1 > a_2$, analogous argument applies.

At (a_1, a_2), by $a_1 \le \overline{a}_1$, $\underline{p}_2 \ge \overline{p}_1 - (a_2 - a_1) = \inf(P_2 \cup P'_2) =: p_2$. Since $p_2 \notin P_2 \cup P'_2$, under any distribution over $P_2 \cup P'_2$ firm 1 can have a profit higher than (recall $a_1 + a_2 = 1$)

$$\pi_1(p_1^F(p_2), p_2) = \frac{1}{2}\left(2 - 2\sqrt{a_1} - \frac{1}{2}(a_2 - a_1)\right)^2 \quad (43)$$

So, we have $\pi_1(p_1^F(p_2), p_2) > a_2^2/2$ if and only if

$$\begin{aligned} 2 - 2\sqrt{a_1} - \frac{1}{2}(a_2 - a_1) &> a_2 \Leftrightarrow \\ \frac{5}{2} - 2\sqrt{1 - a_2} - 2a_2 &> 0, \end{aligned} \quad (44)$$

where the last inequality is an equality for $a_2 = 3/4$, so it is satisfied for $a_2 \in (1/2, 3/4)$.

Symmetric arguments apply to firm 2. Hence, for each $i = 1, 2$ and $\mu_i \in \Delta^H(\widehat{S}_{-i})$, we have $\rho_i(\mu_i) \subseteq S_i(a_i)$. Then, for each $s_i \in \widehat{S}_i$, there is $\mu_i \in \Delta^H(\widehat{S}_{-i})$ such that $\mu_i(\cdot|a_1, a_2)$ justifies only $s_i(a_{-i})$, and for each $a'_{-i} \neq a_{-i}$, the price distribution induced by $\mu_i(\cdot|a_i, a'_{-i})$ justifies only $s_i(a'_{-i})$.

Finally, fix $a_1 \in (\overline{a}_1, \overline{\overline{a}}_1]$, where $\overline{\overline{a}}_1 \simeq 0.38$ has been defined in Section 3.2.2 as the value of a'_1 such that, at $(a'_1, 1 - a'_1)$, $\overline{p}_2 = \underline{p}_2$. At (a_1, a_2), by $a_1 \in (\overline{a}_1, \overline{\overline{a}}_1]$, we have $\overline{p}_2 \ge \underline{p}_2 > \overline{p}_1 - (a_2 - a_1)$. Let $p'_2 := \underline{p}_2$ and $p'_1 := p'_2 + (a'_2 - a'_1) > \overline{p}_1$. Fix $\varepsilon \in (0, p'_1 - \overline{p}_1)$ and let $P_i := [p'_i - \varepsilon, p'_i)$ for each $i = 1, 2$: for sufficiently small δ, each $p_i \in P_i$ is the unique best reply to the uniform distribution over $\left[p'_{-i} - (p'_i - p_i), p'_{-i} - (p'_i - p_i) + \delta\right] \subseteq P_{-i}$ (see Appendix A.1 for the construction of such distributions). Thus, $P_1 \times P_2$ is a best response set, and let $P'_1 \times P'_2$ be the symmetric one. By $p_1^F(p'_2) =$

$p_2' + (a_2 - a_1)$ and $p_2' > \overline{p}_1 - (a_2 - a_1) =: p_2$, for sufficiently small ε we also have $a_1 \cdot (p_2' - \varepsilon + (a_2 - a_1)) > \pi_1(p_1^F(p_2), p_2) > a_2^2/2$, where the last inequality comes from (43) and (44). Then, $\widehat{S}_1 \times \widehat{S}_2$ defined as in (42) is an EFBRS.

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