School of Mathematics and Statistics
Lanzhou University

International Workshop
"Infinite-Dimensional Dynamical Systems and Attractors"
June 30 – July 6, 2019

On the limit dynamics of systems of nonlinear parabolic equations

A.V. Romanov
National Research University Higher School of Economics
Moscow, Russia
(**av.romanov@hse.ru**)

Problem statement

The question:

under what conditions does the final (for a long time) phase dynamics of systems of one-dimensional dissipative reaction-diffusion equations with convection be successfully described by some ODE in $\mathbb{R}^n$?

Talk plan

- ▲ **Abstract semilinear equations**
- ▲ **Inertial manifolds and the finite-dimensionality of limit dynamics on attractor**
- ▲ **Parabolic systems: Dirichlet conditions**
- ▲ **Parabolic systems: Neumann conditions**
- ▲ **Parabolic systems: periodic conditions**

Some necessary backgrounds!

Abstract semilinear parabolic equations

First we consider abstract dissipative semilinear parabolic equations

$$\partial_t u = -Au + F(u) \qquad (*)$$

(SPE) in an infinite-dimensional separable Hilbert space $(X, \|\cdot\|)$. We assume that an unbounded positive definite linear operator A with domain of definition $D(A)$ has compact resolvent. We put $X^\alpha = D(A^\alpha)$ with $\alpha \geq 0$; then we have $\|u\|_\alpha = \|A^\alpha u\|$, $X^0 = X$, and $X^1 = D(A)$. Let a nonlinear function $F \in C^2(X^\alpha, X) \bigcap \mathrm{Lip}(X^\alpha, X)$ for some $\alpha \in [0,1)$; then equation $(*)$ generates a smooth compact resolving semiflow $\{\Phi_t\}_{t\geq 0}$ in the phase space X^α.

D. Henry, "*Geometric theory of semilinear parabolic equations*", Lect. Notes in Math., **840**, Springer, 1981.

The dissipativity of SPEs

The dissipativity of SPE means that

$$\sup \lim_{t \to +\infty} \|\Phi_t u\|_\alpha \le r$$

for some $r > 0$ uniformly in u from bounded subsets $\Omega \subset X^\alpha$. In this case, there exists a **compact global attractor** $\mathcal{A} \subset X^\alpha$ consisting of all bounded complete trajectories $\{u(t)\} \subset X^\alpha, t \in \mathbb{R}$. In fact, thanks to the smoothing effect of parabolic equation we have $\mathcal{A} \subset X^1$.

A.V. Babin and M.I. Vishik, "*Attractors of evolution equations*", North-Holland, 1992.

Inertial manifolds

We say that SPE $(*)$ is asymptotically finite-dimensional if there exists an inertial manifold (IM) for it, i.e., a smooth finite-dimensional surface $\mathcal{M} \subset X^{\alpha}$, containing an attractor and exponentially attracting all trajectories at a large time. The restriction of $(*)$ to $\mathcal{M}$ is an ODE in $\mathbb{R}^n$, $n = \dim \mathcal{M}$, which completely describes the final dynamics of the evolutionary system.

G. Sell and Y. You, "*Dynamics of Evolutionary Equations*", Springer, 2002.

The finite-dimensionality of limit dynamics

A less rigorous approach to the problem of finite-dimensionality of the SPE limit dynamics was proposed in [1].

We say that the **dynamics of ($*$) on the attractor is finite-dimensional** = **SPE ($*$) has FDA property,** if

for some ODE $\partial_t x = g(x)$ *in* $\mathbb{R}^n$ *with* $g \in \mathrm{Lip}(\mathbb{R}^n, \mathbb{R}^n)$ *and resolving flow* $\{\Theta_t\}_{t\in\mathbb{R}}$, *there exists an invariant compact set* $\mathcal{K} \subset \mathbb{R}^n$ *such that the dynamical systems* Φ_t *on* $\mathcal{A}$ *and* Θ_t *on* $\mathcal{K}$ *are Lipschitz conjugate for* $t \geq 0$.

The properties of the dynamics to be asymptotically finite-dimensional or finite-dimensional on the attractor have not still been separated.

The FDA criteria

Let $G(u)=F(u)-Au$ is the vector field of SPE $(*)$ .We list **a few different criteria** for the dynamics to be finite-dimensional on the attractor.

(i) The vector field G is Lipschitz on attractor $\mathcal{A}$:

$$\|G(u)-G(v)\|_\alpha \le C_1\|u-v\|_\alpha .$$

(ii) The phase semiflow on $\mathcal{A}$ expands to the Lipschitz flow:

$$\|\Phi_t(u)-\Phi_t(v)\|_\alpha \le C_2\|u-v\|_\alpha e^{k|t|}, \quad t\in\mathbb{R}.$$

(iii) The attractor is the Lipschitz grapf: there is exist the finite-dimensional projector $P\in \operatorname{End} X^\alpha$ such that

$$\|Pu-Pv\|_\alpha \ge C_3\|u-v\|_\alpha \text{ for } u,v\in\mathcal{A}.$$

(iv) The spherical projection $\mathcal{A}^s = \left\{w\in X^\alpha :\ w=\dfrac{u-v}{\|u-v\|_\alpha}\right\}$,

where $u,v\in\mathcal{A}$, $u\ne v$, is pre-compact in X^α. The positive constants C_l $(1\le l\le 3)$ and k depend here from $\mathcal{A}$ only.

A.V. Romanov, *Sb. Mathematics*, **191**:3, 2000.
A.V. Romanov, *Izv. Math.*, **65**:5 (2001).

The FDA condition

A sufficient condition of FDA property: attractor contained in some smooth finite-dimensional manifold $\mathcal{M} \subset X^{\alpha}$.

Remark. The manifold $\mathcal{M}$ not assumed to be invariant in this condition!

A.V. Romanov, *Izv. Math.*, **65**:5 (2001).

The more constructive FDA conditions

The analytical conditions for the dynamics to be finite-dimensional on the attractor originate from the decomposition

$$G(u)-G(v)=(T_0(u,v)-T(u,v))(u-v) \qquad (1)$$

of differences of the vector field $G(u)=F(u)-Au$ on $\mathcal{A}$, where $T_0(u,v)\in \operatorname{End} X^{\alpha}$ and $T(u,v)\in \operatorname{End}(X^{\alpha},X)$ are unbounded linear operators similar to normal ones. In details:

$$T(u,v)=S^{-1}(u,v)H(u,v)S(u,v)\,, \qquad (2a)$$

where operators $H(u,v)$ are normal in X, and

$$\|T_0(u,v)\|_{\alpha}\le\gamma,\quad \|S(u,v)\|\le\gamma,\quad \left\|S^{-1}(u,v)\right\|\le\gamma,\quad \|\partial_t S(u,v)\|\le\gamma, \qquad (2b)$$

with some $\gamma=\gamma(\mathcal{A})>0$. We denote by $\partial_t S(u,v)$ derivative in zero of the smooth function

$$S(\Phi_t u,\Phi_t v):[0,\infty)\to X\ .$$

The more constructive FDA conditions - 1

We write

$$\Gamma_a = \{z \in \mathbb{C} : \operatorname{Re} z = a\}, \qquad \Gamma(a,\xi) = \{z \in \mathbb{C} : a - \xi \le \operatorname{Re} z \le a + \xi\}$$

for $a > \xi > 0$ and assume that, for some $d > 0$, $\theta \in [0,1]$, the ‘‘combined spectrum’’

$$\Sigma = \bigcup_{u,v} \sigma(T(u,v)), \quad u,v \in \mathcal{A}$$

is localized in the domain

$$\Omega(d,\theta) = \{x + iy \in \mathbb{C} : |y| < dx^{\theta}\}, \quad x > 0.$$

Let $\beta = \alpha/2$ for $0 \le \theta \le \alpha/2$, and let $\beta = (\alpha+\theta)/3$ for $\alpha/2 < \theta \le 1$. We assume that the set $\mathbb{C} \setminus \Sigma$ contains strips $\Gamma(a_\nu, \xi_\nu)$, whereas $a_\nu, \xi_\nu \to \infty$ as $\nu \to +\infty$. It is known [1] that under the condition

$$a_\nu^{\beta} = o(\xi_\nu) \qquad (\nu \to +\infty) \tag{3}$$

and restrictions (2) on the operators $T_0(u,v)$, $T(u,v)$, the dynamics of SPE ($*$) on the attractor is finite. If $A^* = A$, then $\Sigma \subset \Omega(d,\alpha)$ and relation (3) becomes $a_\nu^{2\alpha/3} = o(\xi_\nu)$. For the existence of an inertial manifold in this situation, the more rigid (for $\alpha > 0$) condition $a_\nu^{\alpha} = o(\xi_\nu)$ is required.

[1] A.V. Romanov, *Izv. Math.*, **65**:5 (2001).

Parabolic systems

Now we consider systems of equations of the form

$$\partial_t u = \partial_{xx} u + f(x, u, u_x), \qquad x \in (0,1), \qquad (**)$$

with $u = (u_1, \ldots, u_m)$ and with one of the following boundary conditions:

$$\text{(Dirichlet)}, \quad u(0) = u(1) = 0; \qquad \text{(D)}$$

$$\text{(Neumann)}, \quad u_x(0) = u_x(1) = 0\,; \qquad \text{(N)}$$

$$\text{(periodicity)}, \quad u(0) = u(1), \quad u_x(0) = u_x(1)\,. \qquad \text{(P)}$$

Let $J = [0,1]$ for the cases (D) and (N), and let $J = \mathbb{R} | \operatorname{mod} \mathbb{Z}$ for the case (P). We believe that

$$f \in C^\infty(J \times \mathbb{R}^{2m}, \mathbb{R}^m)\,.$$

In the case of the Dirichlet conditions, $f(x, 0, p) = 0$ for $x = 0,1$ and $p \in \mathbb{R}$.

Parabolic systems - 1

We put $X = L^2(J;\mathbb{R}^m)$ and write system $(**)$ in the form $(*)$ with $Au = u - \partial_{xx}u$ and $F(u) = u + f(x,u,u_x)$. As the phase space, we take X^{α} with $\alpha \in (3/4,1)$; in this case, then $X^{\alpha} \subset C^1(J)$, $X^{\alpha+1/2} \subset C^2(J)$. Further, $I \doteq \mathrm{Id}$ in $\mathbb{R}^m$ or in X and

$$G(u) = \partial_{xx}u + f(x,u,u_x)$$

is the vector field of $(**)$. We assume that system $(**)$ is X^{α}-dissipative and has global attractor $\mathcal{A} \subset X^1$. For example [1], it is true in the cases (D) or (P) and $m = 1$ under the conditions:

$$f(x,u,0)\,\mathrm{sign}\,u \to -\infty \text{ for } |u| \to \infty \text{ uniformly in } x \in J\,;$$

$$|f| + |f_x| + |f_u| \le M(u)(1+p^2), \qquad |f_p| \le M(u)(1+|p|)\,.$$

[1] A.V. Babin and M.I. Vishik, "*Attractors of evolution equations*", North-Holland, 1992.

Parabolic systems: two approaches

The final dynamics of systems $(**)$ with spatial homogeneous nonlinearity $f(u,u_x)$ was studied in papers [1, 2]. It was shown [1] that, under conditions (D) and (N), such systems admit a smooth inertial manifold. In the periodic case, the existence of a smooth IM was established [2] under the condition that the $(m\times m)$ matrix f_p is diagonal with a unique nonzero element.

The results obtained in [1, 2] are based on a change of the phase variable u which ‘‘decreases’’ the dependence of the nonlinearity f on u_x.

In my talk the variable is changed for linearized system $(**)$. In this case, we only obtain that the dynamics is finite-dimensional on the attractor but do not obtain the existence of IM.

At the same time, the spatially inhomogeneous case $f = f(x,u,u_x)$ is also considered. It is also discovered that the limit dynamics is finite-dimensional for one type of systems $(**)$ – (P).

[1] A. Kostianko and S. Zelik, *Comm. Pure Appl. Anal.*, **16**:6 (2017).
[2] A. Kostianko and S. Zelik, *Comm. Pure Appl. Anal.*, **17**:1 (2018).

Decomposition of the vector field on an attractor

Following the format of (1), we put

$$b(x;u,v)=\int_0^1 f_p(x,s,s_x)d\tau, \qquad b_0(x,u,v)=\int_0^1 f_u(x,s,s_x)d\tau,$$

for $u,v\in\mathcal{A}$, $s(x)=\tau u(x)+(1-\tau)v(x)$, $x\in J$, and

$$G(u)-G(v)=R(u,v)h=\partial_{xx}h+b_0(x;u,v)h+b(x;u,v)h_x$$

for $h=u-v$. Proceeding as in [1], to the differential expression

$$Rh=h_{xx}+B_0(x)h+B(x)h_x,$$

where $B(x)=b(x;u,v)$ and $B_0(x)=b_0(x;u,v)$, we apply (for fixed u,v) an analog of the Liouville transformation $h=U\eta$, where the nonsingular matrix function $U(x)$, $x\in J$, is the solution of the Cauchy problem

$$U_x=-\frac{1}{2}B(x)U, \quad U(0)=I.$$

Since $U_{xx}=-\frac{1}{2}(B_x(x)U+B(x)U_x)=-\frac{1}{2}B_x(x)U+\frac{1}{4}B^2(x)U$, we have

$$\begin{aligned}Rh&=RU\eta=U\eta_{xx}+2U_x\eta_x+U_{xx}\eta+B_0(x)U\eta+B(x)(U_x\eta+U\eta_x)\\&=U\eta_{xx}-B(x)U\eta_x-\frac{1}{2}B_x(x)U\eta+\frac{1}{4}B^2(x)U\eta+B_0(x)U\eta+B(x)U\eta_x+B(x)(-\frac{1}{2}B(x)U\eta)\\&=U\eta_{xx}+(B_0(x)-\frac{1}{2}B_x(x)-\frac{1}{4}B^2(x))U\eta=U(U^{-1}h)_{xx}+Qh. \qquad (4)\end{aligned}$$

[1] D.A. Kamaev, *Russ. Math. Surv.*, **47**:5 (1992).

Dirichlet conditions

In the transition from h to η, conditions (D) are preserved. If we set $H_0\eta = -\partial_{xx}\eta$, $\eta(0)=\eta(1)=0$, then $H_0^* = H_0$ in X. Let be $T(u,v) = UH_0U^{-1}$ and $T_0(u,v)h = Q(x)h$ in decomposition (1) of differences of the vector field (**) on the attractor. For the combined spectrum $\Sigma = \{\pi^2\nu^2, \nu \in \mathbb{N}\}$, we have

$$\theta = 0, \quad \beta = \alpha/2 < 1/2, \quad a_\nu = \pi^2(\nu^2 + \nu + 1/2), \quad \xi_\nu = \pi(\nu + 1/2)$$

and $a_\nu^\beta = o(\xi_\nu)$, $\nu \to \infty$. We can show that the norms of the operators $T_0(u,v) \in \operatorname{End} X^\alpha$ and

$$U(u,v),\ U^{-1}(u,v),\ \partial_t U^{-1}(u,v) \in \operatorname{End} X$$

are uniformly bounded by $u, v \in \mathcal{A}$. Then by [1] the dynamics (**) – (D) is finite-dimensional on the attractor.

[1] A.V. Romanov, *Izv. Math.*, **65**:5 (2001).

Neumann conditions

Conditions (N) are not preserved when we pass from h to η, but this case can be reduced to the preceding one by using the technique of [1]. We will embed system $(**)$ – (N) in a large system of special structure. Namely, we differentiate Eqs. $(**)$ with respect to x, put $w = u_x$, and obtain the system of $2m$ equations

$$u_t = u_{xx} + f(x,u,w),$$
$$w_t = w_{xx} + f_x(x,u,w) + f_u(x,u,w)w + f_p(x,u,w)w_x$$

with boundary conditions

$$u_x(0) = u_x(1) = 0, \qquad w(0) = w(1) = 1.$$

The embedding operator is $u \to (u,u_x) = (u,w)$. With a suitable choice of phase space, this system will also be dissipative. Since the first part of the new system does not contain u_x, we can apply the Liouville transformation only in variable w, then boundary conditions (N) – (D) are preserved in the whole. Based on the preceding considerations, we can conclude that the limit dynamics of original system $(**)$ – (N) is finite-dimensional.

[1] A. Kostianko and S. Zelik, Comm. Pure Appl. Anal., **16**:6 (2017).

The dynamics on attractor: (D) and (N) conditions

Thus, we have the following assertion.

THEOREM 1. *If problem* $(**)-$ (D) *or* $(**)-$ (N) *is dissipative in* X^{α} *with* $\alpha \in (3/4, 1)$, *then the phase dynamics on the attractor* $\mathcal{A} \subset X^1$ *is finite-dimensional.*

Periodic conditions

We pass to the case of periodic problem $(**)$ – (P). In this situation, the space X^{α} with $\alpha \in (3/4, 1)$ is the Banach algebra and $B(0) = B(1)$, $B_x(0) = B_x(1)$. We put $V = U^{-1}$, then we have

$$\eta = Vh, \quad \eta_x = V_x h + V h_x, \quad V_x = \frac{1}{2} VB, \quad V(0) = E, \quad V_x(0) = \frac{1}{2} B(0),$$

$$V_x(1) = \frac{1}{2} V(1) B(0),$$

$$\eta(0) = h(0), \quad \eta(1) = V(1) h(1),$$

$$\eta_x(0) = \frac{1}{2} B(0) h(0) + h_x(0), \quad \eta_x(1) = \frac{1}{2} V(1) B(0) h(1) + V(1) h_x(1).$$

As we see, periodic conditions (P) for h turn into the conditions

$$\eta(1) = V(1)\eta(0), \qquad \eta_x(1) = V(1)\eta_x(0). \tag{5}$$

and, generally speaking, the monodromy operator $V(1) \neq I$.

The main lemma

LEMMA 1. *If periodic problem* (**) – (P) *is dissipative in* X^{α} *with* $\alpha \in (3/4, 1)$ *and the operator* $V(1)$ *in* $\mathbb{R}^m$ *is normal, then the phase dynamics on the attractor is finite-dimensional.*

PROOF. Let $\varphi_j \in \mathbb{R}^m$ and $\mu_j = |\mu_j| e^{i\theta_j}$, $\theta_j \in (-\pi, \pi]$, be orthonormal eigenvectors and eigenvalues of the operator $V(1)$ with $1 \le j \le m$. We put $H_0 = H_0(u,v) = \omega I - \partial_{xx}$ for boundary conditions (5) with some $\omega > 0$ and $T = UH_0U^{-1}$. The eigenfunctions and eigenvalues of the operator H_0 have the form

$$\psi_{k,j}(x) = e^{x \ln \mu_j} \cdot e^{2\pi k i x} \cdot \varphi_j, \quad x \in J,\ k \in \mathbb{Z},\ 1 \le j \le m,$$

$$\lambda_{k,j} = \omega - (\ln|\mu_j| + i\theta_j + 2\pi k i)^2 = \omega + (2\pi k + \theta_j - i \ln|\mu_j|)^2. \tag{6}$$

Here $\ln \mu_j = \ln|\mu_j| + i\theta_j$. The quantities $\mu_j = \mu_j(u,v) \ne 0$ continuously depend on $u, v \in \mathcal{A}$, and hence, by compactness of $\mathcal{A}$, we have $0 < c_1 \le |\mu_j| \le c_2$ for some $c_l = c_l(\mathcal{A})$, $l = 1, 2$. Thus, the quantities $|\ln|\mu_j||$ in (6) are uniformly bounded. If $S = \mathrm{diag}(e^{-x \ln \mu_j})$, then the system of functions $S\psi_{k,j}$ is complete and orthonormal in $X = L^2(J; \mathbb{R}^m)$, and the operators $H(u,v) = SH_0S^{-1}$ are normal in X.

The main lemma - 1

Returning to expression (4), we put $T(u,v)=US^{-1}HSU^{-1}$ and $T_0(u,v)=\omega I+Q(u,v)$ in decomposition (1) of differences of the vector field $(**)$ on the attractor. The unbounded operators $T(u,v)$ in X are similar to the normal ones. We choose the parameter $\omega>0$ so as to ensure the inclusion $\Sigma\subset\Omega(d,\theta)$ with $\theta=1/2$ and an appropriate $d>0$. Since $3/4<\alpha<1$, we have $\beta=(\alpha+\theta)/3<1/2$. Moreover, it follows from (6) that the set $\mathbb{C}\setminus\Sigma$ contains the strips $\Gamma(a_k,\xi_k)$ with

$$a_k\sim 4\pi^2k^2,\quad \xi_k\sim 4\pi^2k\,,$$

and therefore, $a_k^{\beta}=o(\xi_k)$. We can show that the norms of the operators

$T_0=T_0(u,v)\in \mathrm{End}\,X^{\alpha}$ and

$$U,\ U^{-1},\ S,\ S^{-1},\ \partial_t(SU^{-1})\ \in \mathrm{End}\,X$$

are uniformly bounded by $u,v\in\mathcal{A}$. Then by [1] dynamics $(**)$ – (P) on the attractor is finite-dimensional.

[1] A.V. Romanov, *Izv. Math.*, **65**:5 (2001).

The dynamics on attractor: periodic conditions

THEOREM 2. *Let system* (**) – (P) *is dissipative in* X^{α} *with* $\alpha \in (3/4, 1)$. *Then the phase dynamics on the attractor is finite-dimensional if the matrix* $f_p(x,u,u_x)$ *is diagonal for* $u \in \mathrm{co}\,\mathcal{A}$*;*

PROOF. By Lemma 1, it suffices to prove the normality of the monodromy operator $V(1)$. We have

$$B(x) = B(x;u,v) = \int_0^1 f_p(x,s,s_x)d\tau$$

for $u, v \in \mathcal{A}$, $s(x) = \tau u(x) + (1-\tau)v(x)$, $x \in \mathbb{R}$. The matrix $B(x)$ is diagonal, and hence, $V(x) = \mathrm{diag}$, $x \in \mathbb{R}$, for the solution of the Cauchy problem

$$V_x = \frac{1}{2}VB, \quad V(0) = I.$$

So the operator $V(1)$ is self-adjoint and the theorem is proved.

What may be done in future

The problem: to characterize the widest possible class of periodic problems $(**)$ for which the dynamics on the attractor is finite-dimensional.

The counterexample [1] shows that for such systems this property of phase dynamics is not always satisfied.

[1] A. Kostianko and S. Zelik, *Comm. Pure Appl. Anal.*, **17**:1 (2018).

The main references

- A.V. Romanov, “Finite-dimensional limit dynamics of dissipative parabolic equations”, *Sb. Mathematics*, **191**:3, 2000.
- A.V. Romanov, “Finite-dimensionality of dynamics on an attractor for non-linear parabolic equations”, *Izv. Math.*, **65**:5 (2001).
- Kostianko and S. Zelik, “Inertial manifolds for 1D reaction-diffusion-advection systems. Part I: Dirichlet and Neumann boundary conditions”, *Comm. Pure Appl. Anal.*, **16**:6 (2017).
- Kostianko and S. Zelik, “Inertial manifolds for 1D reaction-diffusion-advection systems. Part II: Periodic boundary conditions”, *Comm. Pure Appl. Anal.*, **17**:1 (2018).
- D.A. Kamaev, “Families of stable manifolds of invariant sets of systems of parabolic equations”, *Russ. Math. Surv.*, **47**:5 (1992) .

THANKS FOR YOUR ATTENTION !