A simple solution to the Hotelling problem

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Abstract

I show that in the original two-stage Hotelling model with linear transportation cost, the transportation-efficient location pair (1/4, 3/4) is the only symmetric pair induced by a pure Nash equilibrium in backwards-rationalizable strategies (Penta 2015, Perea 2014). This solution is supported by natural, expressible threats that are also compatible with forward induction reasoning, and remains unique when firms can also agree on a *Self-Enforcing Set* of strategy profiles (Catonini, 2020b).

Keywords: Hotelling, backward induction, forward induction, self-enforcing agreements.

1 Introduction

The solution of the Hotelling problem in its original formulation has been a long-standing issue in economics. Hotelling (1929) predicted that the two firms, in the attempt to acquire a competitive advantage before the pricing stage, would converge to the middle of the spectrum — the so-called principle

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of minimum differentiation. Many decades later, d'Aspremont et al. (1979) found a mistake in Hotelling's argument. Hotelling did not consider that, if the firms get too close to each other, they have the incentive to *undercut* any sufficiently high price of the competitor by more than the transportation cost between their locations, so to conquer the whole market. Because of this tendency, there is no subgame perfect equilibrium in pure strategies. To obtain one, D'Aspremont et al. make a radical modification to the model: they impose a quadratic instead of linear transportation cost. This makes serving far-away customers increasingly expensive in terms of size of the necessary undercut. With this, they obtain a unique equilibrium in every subgame and a unique subgame perfect equilibrium for the whole game, where the firms locate at the opposite extremes of the spectrum — the so-called principle of maximum differentiation. Economides (1986) has observed that this extreme result is really an artifact of the convexity of the transportation cost: the firms will move a bit closer to the middle when the convexity is reduced. However, Economides' analysis is constrained by the focus on subgame perfect equilibrium in pure strategies, which soon ceases to exist as convexity diminishes. Osborne and Pitchik (1987) go back to linear transportation cost and look for a subgame perfect equilibrium with numerical methods. They find that firms should locate around $(0.27, 0.73)$, where there is no pure pricing equilibrium. Less confidently, they also speculate that the subgame perfect equilibrium is unique.

Subgame perfect equilibrium is traditionally regarded as the natural extension of Nash equilibrium to multistage games, especially when one wants to preserve the self-enforceability of the solution. However, there are at least two possible critiques of this view. First, it is not clear why players should still believe in the designated equilibrium after a unilateral deviation. A voluntary deviation displays disbelief of some feature of the overall equilibrium, most likely of the threat that is supposed to deter the deviation. Second, when the only subgame perfect equilibrium requires randomization, players lack the strict incentive, or even the ability, to actually randomize with the required odds and stick to the outcome of the randomization. This seems to go against the very spirit of self-enforceability.[1]

These two difficulties emerge clearly in the Hotelling model. Firms are rational agents whose deviations cannot be disregarded as non-strategic. Moreover, in the numerical solution of Osborne and Pitchik, firms randomize over prices on path. An additional related difficulty is that the required randomization is complicated and could only be retrieved with numerical methods, so it hardly approximates a plausible behavior of real firms. Note also that the self-enforceability of the locations relies on very high confidence in the exact randomization over prices: a small unilateral deviation outwards would kill any uncertainty about prices and be profitable against the vast majority of randomizations over the rationalizable or equilibrium prices at the expected locations. It is also striking that a small joint deviation to $(1/4, 3/4)$ would restore uniqueness of rationalizable prices as well, with a Pareto improvement of firms' profits, hence appearing as a much more plausible self-enforcing agreement between the two firms.

Starting from these considerations, I investigate whether different principles than subgame perfection can yield a more satisfactory solution to the Hotelling problem. I maintain that firms coordinate on a path, or a set of paths, but only formulate unilateral threats, and do not neutralize the residual uncertainty with a probability distribution. Mathematically, this translates into Nash equilibria of the strategic form, or *Self-Enforcing Sets* (Catonini 2020b).[2] Moreover, I challenge the credibility of threats with strategic reasoning. To start, I assume that firms engage in backward induction reasoning. This only requires common belief that firms choose rational prices in the subgame; that is, a price that best replies to some conjecture about the competitor's price. I capture backward induction reasoning with backwards rationalizability (Penta 2015, Perea 2014). Forward induction reasoning relies

[1] Of course, randomizations can find alternative interpretations in the context of a population game, but this is very far from the strategic context of an oligopoly.

[2] The original definition of Self-Enforcing Set already subsumes a form of forward induction reasoning. It does so by requiring the strategies in the set to be extensive-form rationalizable. Here I start from the plain-vanilla notion of Self-Enforcing Set and restrict the realm of strategies with different forms of strategic reasoning.

on the belief that a deviation from the expected location and the subsequent price be part of the same rational plan. This idea has been made precise with many different nuances, corresponding to different strategic reasoning hypotheses. Here I consider players who rationalize deviations under the view that the deviator did believe in the coordinated path (i.e., that the competitor would have complied with it), but does not believe in the threat. This means that the deviator is expected to aim for a higher payoff than under the coordinated path, in line with the forward induction refinements of equilibrium.[3]

I find that the transportation-efficient location pair $(1/4, 3/4)$ is the only symmetric location pair induced by a Nash equilibrium in backwards rationalizable strategies. This solution survives the forward induction refinement, and remains unique when one considers all Self-Enforcing Sets in backwards rationalizable strategies. The intuition is straightforward. At locations pairs (a_1, a_2) with $a_1 \leq 1/4$ and $a_2 \geq 3/4$, there is a unique pure pricing equilibrium, which is also the only rationalizable price pair. Given the location of the competitor, the closer a firm is to the middle, the higher its equilibrium profit. Therefore, if the firms were to coordinate on a location pair $(a_1, 1 - a_1)$ with $a_1 < 1/4$, each firm could profitably relocate towards the middle. At locations pairs $(a_1, a_2) \neq (1/4, 3/4)$ with $a_1 \geq 1/4$ and $a_2 \leq 3/4$ there is no pure pricing equilibrium, and the rationalizable prices include undercuts. Therefore, if the firms were to agree on a locations pair $(a_1, 1 - a_1)$ with $a_1 \in (1/4, 1/2)$, they would also have to agree on a set of prices closed under rational behavior (Basu and Weibull, 1991) that entails the possibility of undercuts. Then, in case of pessimistic belief over this set, each firm would rather "give in" and move outwards, to a location where undercutting is no more rationalizable for the competitor. At $(1/4, 3/4)$, there is a unique rationalizable and equilibrium price pair. A deviation towards the middle, instead, entails uncertainty over prices. In particular, the deviator has the incentive to undercut any price of the competitor that makes the deviation profitable. In turn, the competitor has the incentive to respond with a low price, that the deviator has no incen-

[3]E.g.: strategic stability (Kohlberg and Mertens, 1986), intuitive criterion (Cho and Kreps 1987), divine equilibrium (Banks and Sobel 1987).

tive to undercut. Hence, the location pair $(1/4, 3/4)$ is sustained by a very natural imprecise threat: "if you deviate towards the middle, I will make sure you will not undercut me!" The corresponding Self-Enforcing Set can be interpreted as a self-enforcing agreement between the two firms, where, arguably, the natural, conservative reactions to deviations may not even need to be made explicit for deterrance.

The paper is organized as follows. In Section 2 I introduce some well-known facts and relevant prices for the pricing subgame, backwards rationalizability, and Self-Enforcing Sets. In Section 3 I prove existence, and in Section 4 I prove uniqueness of the desired solution. The Online Appendix shows existence and uniqueness of the solution under the original definition of Self-Enforcing Set of Catonini (2020b), which uses extensive-form rationalizability in place of backwards rationalizability.[4]

2 Preliminaries

Model Two firms, $i = 1, 2$, sell the same good at locations a_1 and a_2 of a continuum of buyers of measure 1.[5] There is no cost of production. Every buyer buys one unit from one of the two firms, except when both prices are "prohibitively high" — the precise level is immaterial for the analysis. The payoff of buyer $j \in [0, 1]$ when she buys from firm i is $-p_i - |j - a_i|$, where p_i is the price fixed by firm i. The buyer chooses at random one of the two firms when indifferent. The two firms first choose simultaneously a_1 and a_2, and then, after observing (a_1, a_2), fix prices simultaneously.

Optimal, equilibrium, and rationalizable prices Fix $a = (a_1, a_2)$. If $a_1 = a_2$, there is Bertrand competition and the unique equilibrium and

[4] In Catonini (2020b), as an example, I solve a discretized and simplified version of the model with this notion of Self-Enforcing Set.

[5] This parametrization of the model corresponds to the one of Osborne and Pitchik (1987), i.e., to the choice of $c = 1$ and $l = 1$ in d'Aspremont et al. (1979). Note: in those two papers the position of firm 2 is identified by the distance $b = 1 - a_2$ from buyer $j = 1$.

rationalizable price pair is $(0,0)$. Else, suppose without loss of generality that $a_1 < a_2$. Let $\Delta^a := a_2 - a_1$. Then, firm 1 faces demand

$$D_1(p_1, p_2) = \begin{cases} 0 & \text{if } p_1 - p_2 > \Delta^a \\ \frac{a_1}{2} & \text{if } p_1 - p_2 = \Delta^a \\ \frac{a_1+a_2}{2} + \frac{p_2-p_1}{2} & \text{if } |p_1 - p_2| < \Delta^a \\ a_2 + \frac{1-a_2}{2} & \text{if } p_2 - p_1 = \Delta^a \\ 1 & \text{if } p_2 - p_1 > \Delta^a \end{cases}.$$

Fix p_2. Suppose that p_1 is such that $|p_1 - p_2| < \Delta^a$. Then, the profit of firm 1 is

$$\pi_1(p_1, p_2) = p_1 \left(\frac{a_1 + a_2}{2} + \frac{p_2 - p_1}{2} \right). \tag{1}$$

The first-order condition yields

$$p_1^F(p_2) = \frac{a_1 + a_2}{2} + \frac{p_2}{2}. \tag{2}$$

For $p_1^F(p_2)$ to be the best reply to p_2, two conditions must be verified. First, that $p_1^F(p_2) - p_2 < \Delta^a$. This yields

$$p_2 > 3a_1 - a_2 =: \underline{p}_2$$

Second, firm 1 should not make a higher profit by taking the whole market with a price slightly below $p_2 - \Delta^a$. The supremum of firm 1's revenues with $p_1 < p_2 - \Delta^a$ is $p_2 - \Delta^a$. Hence, we must have

$$\pi_1(p_1^F(p_2), p_2) \geq p_2 - \Delta^a,$$

which yields

$$p_2 \leq 4 - a_2 - a_1 - 4\sqrt{1 - a_2} =: \overline{p}_2. \tag{3}$$

This also implies $p_1^F(p_2) - p_2 > a_1 - a_2$.[6] So, if $\underline{p}_2 < p_2 \leq \overline{p}_2$, $p_1^F(p_2)$ is the best reply to p_2, and substituting (2) into (1), the optimal profit reads

$$\pi_1(p_1^F(p_2), p_2) = \frac{1}{2}\left(p_1^F(p_2)\right)^2. \tag{4}$$

In all other cases there is no best reply to p_2. If $p_2 > \overline{p}_2$, firm 1 prefers to fix p_1 slightly below $p_2 - \Delta^a$. If $p_2 < \overline{p}_2$ but also $p_2 \leq \underline{p}_2$, firm 1 prefers to fix p_1 slightly below $p_2 + \Delta^a$. If $\overline{p}_2 \leq \underline{p}_2$, there is $\overline{p}_2' \leq \overline{p}_2$ such that firm 1 prefers to fix p_1 slightly below $p_2 - \Delta^a$ if $p_2 > \overline{p}_2'$, and slightly below $p_2 + \Delta^a$ if $p_2 < \overline{p}_2'$. Overall, let $\widehat{p}_2 := \min\{\overline{p}_2, \overline{p}_2'\}$ be the smallest price that firm 1 has no incentive to undercut.

The analogous conditions for firm 2 given p_1 can be obtained by substituting a_1 with $1 - a_2$ and a_2 with $1 - a_1$. So we have

$$\begin{aligned} p_2^F(p_1) \quad &: \quad = 1 - \frac{a_1 + a_2}{2} + \frac{p_1}{2}, \\ \underline{p}_1 \quad &: \quad = 2 - 3a_2 + a_1, \\ \overline{p}_1 \quad &: \quad = 2 + a_1 + a_2 - 4\sqrt{a_1}. \end{aligned}$$

Note that, if $a_1 + a_2 \geq 1$, $\overline{p}_1 \leq \overline{p}_2$ and $\underline{p}_1 \leq \underline{p}_2$.

While firm 1 has no best reply to any $p_2 > \widehat{p}_2$ or $p_2 \leq \underline{p}_2$, a sequence of prices converging to p_2 from above clearly supports a probability distribution with best reply $p_1 = p_2 - \Delta^a$ or $p_1 = p_2 + \Delta^a$: just assign countably many probability values to a subsequence sufficiently concentrated close to p_2, so that a marginal increase of p_1 would entail a proportionally higher loss of expected demand.[7]

Given any conjecture ν with mean $\widetilde{p}_2$, by linearity in p_2, the expected

[6] To see this, rewrite $p_2 \leq \overline{p}_2$ as $p_2 \leq 3a_2 - a_1 + 4(1 - a_2) - 4\sqrt{1 - a_2}$, which implies $p_2 < 3a_2 - a_1$ when $a_2 \neq 1$ or $p_2 < \overline{p}_2$, which in turn is equivalent to $p_1^F(p_2) - p_2 > a_1 - a_2$.
When $a_1 = 1$, we have $p_1^F(\overline{p}_2) = \overline{p}_2 - \Delta^a$, so $p_2 - \Delta^a$ best replies to any $p_2 \geq \overline{p}_2$, because there is no market segment to take by undercutting. This is an (innocuous) exception to the inexistence of best replies discussed below.

[7] This can be obtained because firm 1 loses at once all the customers between a_2 and 1 or between 0 and a_1. When $a_1 = 0$ we have $\underline{p}_2 < 0$, so the case $p_2 \leq \underline{p}_2$ does not materialize, and recall that when $a_2 = 1$, $p_2 - \Delta^a$ is the best reply to any $p_2 > \overline{p}_2$.

revenues of firm 1 over the interval $(\sup(\mathrm{Supp}\nu) - \Delta^a, \inf(\mathrm{Supp}\nu) + \Delta^a)$ are given by (1) with $\widetilde{p}_2$ in place of p_2, thus are increasing below and decreasing above $p_1^F(\widetilde{p}_2)$.

The only candidate pure pricing equilibrium is $(p_1^*, p_2^*) = \big(p_1^F(p_2^*), p_2^F(p_1^*)\big)$; that is

$$(p_1^*, p_2^*) = \left(\frac{2 + a_1 + a_2}{3}, \frac{4 - a_1 - a_2}{3}\right).$$

It is an equilibrium if and only if $p_1^* \leq \overline{p}_1$ and $p_2^* \leq \overline{p}_2$ — when these conditions are verified, we also have $p_1^* > \underline{p}_1$ and $p_2^* > \underline{p}_2$. Let $\widehat{a}_1 \approx 0.30$ be the value of a_1 such that $p_1^* = \overline{p}_1$ at $(a_1, 1)$. It is useful to record that (p_1^*, p_2^*) is an equilibrium if and only if $a \in [0, \widehat{a}_1]$ and $a_2 \in [\widehat{a}_2(a_1), 1]$, where $\widehat{a}_2(a_1)$ is defined by $p_1^* = \overline{p}_1$. This includes all the location pairs $a \in [0, 1/4] \times [3/4, 1]$ and excludes all the location pairs $a \in [1/4, 3/4]^2$ different from $(1/4, 3/4)$.

For any subset of firm 2's prices P_2, let $\Delta(P_2)$ denote the set of Borel probability measures ν such that $\nu(B) = 1$ for some Borel subset $B \subseteq P_2$. The rationalizable prices are given by the following recursive procedure. For each $i = 1, 2$ and $p \geq 0$, let $p \in P_i^1$ if and only if p best replies to some conjecture, considering also that it should not be prohibitively high. For each $n > 1$, $i = 1, 2$, and $p \geq 0$, let $p \in P_i^n$ if and only if it best replies to some $\nu \in \Delta(P_{-i}^{n-1})$. For each $i = 1, 2$, let $P_i^\infty = \cap_{n>0} P_i^n$: these are the rationalizable prices of firm i at (a_1, a_2).

Strategies, Self-Enforcing Sets, and Backwards Rationalizability A strategy is a function $s_i : \{\emptyset\} \cup [0, 1]^2 \to [0, \infty)$, where $s_i^\emptyset := s_i(\emptyset) \in [0, 1]$ is the prescribed location, and $s_i(a_i, a_{-i})$ is the price prescribed for the subgame with locations $(a_i, a_{-i}) \in [0, 1]^2$. Let S_i denote the set of all strategies. Let $S_i(a_i)$ denote the set of all $s_i \in S_i$ with $s_i^\emptyset = a_i$. For any $\overline{S}_{-i} \subseteq S_{-i}$, let $\Delta(\overline{S}_{-i})$ denote the set of Borel probability measures ν over S_{-i} such that $\nu(B) = 1$ for some Borel subset $B \subseteq \overline{S}_{-i}$. Let $r_i(\nu)$ denote the (possibly empty) set of strategic-form best replies to ν, that is, the set of strategies that maximize i's expected revenues under ν.

Here I start from a plain-vanilla version of Self-Enforcing Set (Catonini 2020b) that does not incorporate yet strategic reasoning requirements. Moreover, I will focus on sets of strategy profiles that induce the same, symmetric location pair. So, I say that a set $S^* \subset S_1(a_1) \times S_2(a_2)$ inducing location pair $a = (a_1, a_2)$ is a *Self-Enforcing Set* if for each $i = 1, 2$,

1. for each $s_i \in S_i^*$, there is $\nu \in \Delta(S_{-i}^*)$ such that $s_i \in r_i(\nu)$;
2. for each $\nu \in \Delta(S_{-i}^*)$ and $s_i \in r_i(\nu)$, $s_i^\emptyset = a_i$ and $s_i(a) = s_i^*(a)$.for some $s_i^* \in S_i^*$.

I say that a Nash equilibrium $s^* = (s_1^*, s_2^*)$ inducing location pair $a = (a_1, a_2)$ is *strict* when for every $s_i \in r_i(s_{-i}^*)$, $s_i^\emptyset = a_i$ and $s_i(a) = s_i^*(a)$. Note that a strategy profile s^* is a strict Nash equilibrium if and only if the singleton $\{s^*\}$ is a Self-Enforcing Set.

Backwards rationalizability is the iterated elimination of strategies that, from any point on, are suboptimal under the belief that the opponent will follow one of the strategies that survived the previous step of elimination. For the game at hand, backwards rationalizability can be formalized as follows.

- For each $i \in I$, let $S_i^0 = S_i$.
- For each $n > 0$, $i \in I$, and $s_i \in S_i^{n-1}$, let $s_i \in S_i^n$ if and only if:
  - there is $\nu \in \Delta(S_{-i}^{n-1})$ such that $s_i \in r_i(\nu)$;
  - for each $a \in [0, 1]^2$, $s_i(a)$ best replies to some conjecture over the prices p_{-i} such that $p_{-i} = s_{-i}(a)$ for some $s_{-i} \in S_{-i}^{n-1}$.
- For each $i \in I$, let $S_i^\infty := \cap_{n>0} S_i^n$ be the set of backwards-rationalizable strategies.

Note that, at each step of reasoning and at each location pair, the surviving strategies prescribe only prices that best reply to a conjecture over the prices prescribed by the strategies of the competitor at the previous step. Then, the backwards-rationalizable strategies can prescribe only rationalizable prices of the static pricing game.

3 Solution

3.1 Existence

In this section I show the existence of a strict Nash equilibrium $s^* = (s_1^*, s_2^*) \in S^\infty$ inducing locations $a^* = (a_1^*, a_2^*) = (1/4, 3/4)$. I will rely on the existence of rationalizable prices at all locations, with the property of being low enough after unilateral deviations from the Nash locations. In particular, I show that there is a continuum of rationalizable prices at all location pairs where there is no pure equilibrium, which generates best replies below each $\overline{p}_i$ given the undercutting incentives. From now on, when not clear from the context, I will let $\overline{p}_i(a), p_i^*(a)$,etc. indicate prices at location pair a.

Claim 0: At every $a = (a_1, a_2)$ there is a rationalizable price pair $(p_1^r(a), p_2^r(a))$ so that, for each $i = 1, 2$, $p_i^r(a) = 1$ if $a = a^*$, and $p_i^r(a) < \overline{p}_i(a)$ if $a_i \in \{a_1^*, a_2^*\}$ and $a_{-i} \notin \{a_1^*, a_2^*\}$.

Proof of Claim 0. If $a_1 = a_2$, then $(0, 0)$ is a rationalizable price pair. Now suppose without loss of generality that firm 1 is located to the left of firm 2 and not farther from the center. At every location pair $a = (a_1, a_2)$ where (p_1^*, p_2^*) is an equilibrium, it is of course rationalizable. Given $a_1 + a_2 \geq 1$, when $a_1 \leq 1/4$, (p_1^*, p_2^*) is an equilibrium, and when $a_1 = 1/4$ and $a_2 > 3/4$, we have $p_2^* < \overline{p}_2$, as required. When there is no equilibrium, we must have $\overline{p}_1 < p_1^*$: if we had $p_1^* \leq \overline{p}_1$, since firm 1 is not farther from the center and thus $p_2^* \leq p_1^*$ and $\overline{p}_1 \leq \overline{p}_2$, we would also have $p_2^* \leq \overline{p}_2$, hence equilibrium. In this case, let $p_2^L := \widehat{p}_1 - \Delta^a$ and proceed as follows.

Suppose first that $p_2^L \leq \underline{p}_2$. I am going to show that, for some $\varepsilon > 0$, there is a best response set $P_1 \times P_2$ with $P_1 \supseteq [\widehat{p}_1, \widehat{p}_1 + \varepsilon)$ and $P_2 \supseteq \left(p_2^L, p_2^L + \varepsilon\right)$. Then, by $\widehat{p}_1 \leq \widehat{p}_2 \leq \overline{p}_2$, there are rationalizable prices of firm 2 below $\overline{p}_2$, and when $a_1 = 3/4$, by $\overline{p}_1 > \widehat{p}_1$ (recall $a_2 > a_1$), there are also rationalizable prices of firm 1 below $\overline{p}_1$. Every $p_2 > p_2^L$ best replies to a distribution concentrated above $p_2 + \Delta^a$. If $p_2^L < \underline{p}_2$, fix $\varepsilon > 0$ such that $p_2^L + \varepsilon < \underline{p}_2$. Then, every $p_1 \in [\widehat{p}_1, \widehat{p}_1 + \varepsilon)$ best replies to a distribution concentrated above $p_1 - \Delta^a < p_2^L + \varepsilon$. So, $[\widehat{p}_1, \widehat{p}_1 + \varepsilon) \times \left(p_2^L, p_2^L + \varepsilon\right)$ is a best response set. Suppose now that $p_2^L = \underline{p}_2$. Still, $\widehat{p}_1$ best replies to a distribution concentrated above p_2^L. By $\widehat{p}_1 > \underline{p}_2 \geq \underline{p}_1$, $p_2^H := p_2^F(\overline{p}_1)$ best replies to $\widehat{p}_1 = \overline{p}_1$. Fix $\varepsilon \in (0, \Delta^a)$ such that for every $p_2 \in \left(p_2^L, p_2^L + \varepsilon\right)$, every conjecture on $\left\{p_2, p_2^H\right\}$ with mean $\widetilde{p}_2 < \left(p_1^F\right)^{-1}(p_2 + \Delta^a)$ has best reply $p_1^F(\widetilde{p}_2)$. Such ε exists because as p_2 approaches $p_2^L = \underline{p}_2$, $\left(p_1^F\right)^{-1}(p_2 + \Delta^a)$ approaches p_2, so the probability on p_2^H approaches 0. Then, $[\widehat{p}_1, \widehat{p}_1 + \varepsilon) \times \left(\left(p_2^L, p_2^L + \varepsilon\right) \cup \left\{p_2^H\right\}\right)$ is a best response set.

Now suppose that $p_2^L > \underline{p}_2$. I will show that each firm has an interval of rationalizable prices. There must be a rationalizable price of firm 2 below $\overline{p}_2$, because for every interval P_2 with $\inf P_2 \geq \overline{p}_2 \geq \overline{p}_1$, there is an interval of best replies P_1 with $\inf P_1 = \inf P_2 - \Delta^a \geq p_2^L$, and in turn, by $p_2^L > \underline{p}_2 \geq \underline{p}_1$, an interval of best replies P_2' with $\inf P_2' < \inf P_2$.

I am going to show that, for some $\varepsilon > 0$, firm 1 has an interval of length ε at every step n. Thus, partitioning $[0, p_1^+]$, where p_1^+ is a prohibitive price, into a finite number of intervals of length smaller than $\varepsilon/2$, at least one of these intervals must survive all steps. Every interval generates an interval of best replies, so we obtain a rationalizable interval of firm 2.

For the reasons above, for each $n > 0$, there is an interval $P_{2,n} \subset \left(p_2^L, \overline{p}_2\right) \cap P_2^n$,[8] and we can choose $P_{2,n+1} \subseteq P_{2,n}$. Let $\overline{P}_{2,n}$ be the closure of $P_{2,n}$. Fix $p_2 \in \cap_{n>0} \overline{P}_{2,n}$ (it exists by Cantor's intersection theorem) and let $p_1 := p_1^F(p_2)$ be its best reply. If $p_1 > \overline{p}_1$, let $p_2' := p_1 - \Delta^a$; by $p_2 \neq \underline{p}_2$, we have $p_2' \neq p_2$. If $p_1 \leq \overline{p}_1$, let $p_2' := p_2^F(p_1)$; by $\overline{p}_1 \neq p_1^*$ we have $p_2' \neq p_2$, and by $p_1 > \underline{p}_1$,[9] we have $p_2' < p_1 + \Delta^a$. Then, since $|p_1 - p_2| < \Delta^a$, for small enough $\delta > 0$ the best reply to the conjecture ν with $\nu(p_2) = 1 - \delta$ and $\nu(p_2') = \delta$ is still $p_1^F(\mathbb{E}_\nu(\cdot))$.[10] Then, by continuity of all the involved variables, there are $\varepsilon > 0$ and a neighbourhood O of p_2 (a left neighbourhood if $p_2 = \overline{p}_2$) where this construction yields an interval of best replies to two-points conjectures of length at least ε, no matter which $\widetilde{p}_2 \in O$ in place of p_2 we start from. For every $n > 0$, there always

[8] Starting below p_2^L may first bring (in two steps) above $\overline{p}_2$, but then the argument applies.

[9] By $a_1 + a_2 \geq 1$ and $p_2 > \underline{p}_2$, we have $p_1 > \underline{p}_1$.

[10] Note that the perturbation of p_2 with small probability on p_2' moves $p_1^F(\mathbb{E}_\nu(\cdot))$ in the direction of p_2', so we have $\left|p_2' - p_1^F(\mathbb{E}_\nu(\cdot))\right| < \Delta^a$. If $a_2 = 1$ and $p_2 = \overline{p}_2$, we have $p_1 = p_2 - \Delta^a$; in this case, the best reply to ν is not $p_1^F(\mathbb{E}_\nu(\cdot))$, but is between p_1 and $p_1^F(\mathbb{E}_\nu(\cdot))$, and the argument goes through.

exists $\widetilde{p}_2$ in the interior of $O \cap P_{2,n}$, thus a neighbourhood of $\widetilde{p}_1 := p_1^F(\widetilde{p}_2)$ in P_1^n. Then, not only $\widetilde{p}_2' := p_2^F(\widetilde{p}_1)$ when $\widetilde{p}_1 \leq \overline{p}_1$, but also $\widetilde{p}_2' := \widetilde{p}_1 - \Delta^a$ when $\widetilde{p}_1 > \overline{p}_1$ is in P_2^n, because it best replies to a conjecture concentrated above $\widetilde{p}_1$. Thus, the ε-length interval of best replies to conjectures over $\{\widetilde{p}_2, \widetilde{p}_2'\}$ is in P_1^n. ■

Now I am ready to prove the existence result.

Theorem 1 *There exists a strict Nash equilibrium in backwards-rationalizable strategies that prescribes locations* $(1/4, 3/4)$.

Proof. For each $i = 1, 2$, define s_i^* as $s_i^{*,\emptyset} = a_i^*$ and $s_i^*(a) = p_i^r(a)$ for each $a \in [0,1]^2$. To see that $s^* = (s_1^*, s_2^*)$ is a strict Nash equilibrium, take the viewpoint of firm 1. At a^*, p_1^* is the only best reply to p_2^*, and it brings revenues 1/2. For each $a_1 < 3/4$, firm 1's revenues at $(a_1, 3/4)$ against $\overline{p}_2 = 5/4 - a_1$ are bounded above by $\overline{p}_2 - (3/4 - a_1) = 1/2$, so they are strictly below 1/2 when $p_2 < \overline{p}_2$. For each $a_1 > 3/4$, we have $\overline{p}_2 < 5/4 - a_1$ because firm 2 is to the left of firm 1 and closer to the center, so the revenues against any $p_2 < \overline{p}_2$ are strictly below $5/4 - a_1 - (a_1 - 3/4) < 1/2$ as well. There remains to show that s^* is backwards rationalizable. For each $n > 0$ and $i = 1, 2$, let

$$S_{i,n}^* := \{s_i \in S_i(a_i^*)\,|(s_i(a^*) = 1) \wedge (\forall a \neq a^*, s_i(a) \in P_i^n(a))\}.$$

Let $S_{i,\infty}^* := \cap_{n>0} S_{i,n}^*$. We have $s_i^* \in S_{i,\infty}^*$. So, there only remains to show that $S_{1,\infty}^* \times S_{2,\infty}^* \subset S^\infty$. Suppose by way of induction that, for each $i = 1, 2$, $S_{i,n}^* \subset S_i^n$. Fix $i = 1, 2$ and $s_i \in S_{i,n+1}^*$. Since $s_i^\emptyset = a_i^*$ and $s_i(a^*) = 1$, we have $s_i \in r_i(s_{-i}^*)$, and by the induction hypothesis, $s_{-i}^* \in S_{-i}^n$. For each $a \neq a^*$, $s_i(a) \in P_i^{n+1}(a)$, so $s_i(a)$ best replies to some $\eta \in \Delta(P_{-i}^n(a))$, and for each $p_{-i} \in P_{-i}^n(a)$ we have $p_{-i} = s_{-i}(a)$ for some $s_{-i} \in S_{-i,n}^* \subset S_{-i}^{n-1}$. Hence, $s_i \in S_i^{n+1}$. ■

As clear from the proof of Theorem 1, each firm i can have the incentive to deviate from a_i^* only if the competitor reacts with a price above $\overline{p}_{-i}$.[11] Any

[11] There is no strict incentive to deviate also if the deviator reacts with $\overline{p}_{-i}$, which is

backwards-rationalizable price above $\overline{p}_{-i}$ can be undercut by some backwards-rationalizable price of the deviator.[12] Then, the most interesting Self-Enforcing Set is probably the one that corresponds to the very natural reaction of avoiding being undercut by the deviator.

Remark 1 *Consider the set of backwards-rationalizable strategy profiles* (s_1, s_2) *that prescribe locations* $(a_1^*, a_2^*) = (1/4, 3/4)$*, prices* $(p_1^*, p_2^*) = (1, 1)$ *at* (a_1^*, a_2^*)*, and, after a unilateral deviation to* $a_{-i} \neq a_{-i}^*$*, a price* $s_i(a_i^*, a_{-i}) < \overline{p}_i(a_i^*, a_{-i})$*. This set is a (non-empty) Self-Enforcing Set.*

Forward induction refinement Suppose that firm 2 interprets a unilateral deviation from a_1^* as follows: "Firm 1 did expect me to play a_2^*, and then p_2^* at (a_1^*, a_2^*). Now, if the deviation is rational, firm 1 must believe (in expectation) that I will react with a price above $\overline{p}_2$, otherwise she cannot gain from the deviation." Given this interpretation, which prices can firm 2 expect firm 1 to fix next? One possibility is always that firm 1 will fix a price p_1 that undercuts some $p_2 > \overline{p}_2$.[13] But then, firm 2 has the incentive to react to p_1 with some $p_2' < p_2$, thus with a lower price than firm 1 wished for. Iterating, this guarantees that prices below $\overline{p}_2$ are compatible with this kind of forward induction reasoning.

If every conjecture had a best reply, this argument could be formalized with Strong-Δ-Rationalizability (Battigalli 2003, Battigalli and Siniscalchi 2003)

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the highest price that firm i may not have the incentive to undercut. So, another way to express the threat is essentially the following: "I will fix a price that you have no incentive to undercut!"

[12] Again, while very intuitive, the proof of this fact is not straightforward because there is no best reply to a price $p_{-i} > \overline{p}_{-i}$. The best response sets of prices constructed in the Supplemental Appendix can be used to show this point — the complete argument is available upon request.

[13] When $a_1 < a_2^*$, there is also the possibility that firm 1 fixes a price $p_1 > p_1^F(\overline{p}_2)$, but also in this case firm 2 would react with $p_2 < \left(p_1^F\right)^{-1}(p_1)$ (in the interesting case of $a_1 > a_1^*$, by undercutting p_1). This suggests that *all* the off-path belliefs compatible with forward induction reasoning could sustain the desired path, but such stronger criterion was not adopted for the following reason: deviations to $a_1 \in (a_2^*, 1]$, albeit seemingly less sensible, do not have this property.

under the first-order belief restriction that the competitor does not deviate from the expected path. However, at some step of Strong-Δ-Rationalizability, for some deviation a_1, this argument could reduce a firm's price to just one point above $\overline{p}_i$, hence the competitor would not have best replies and the empty set would obtain. This is undesirable: after all, firms must make a choice even when an optimal one does not exist.

To circumvent this problem, one could use ε-best replies and show that Strong-Δ-Rationalizability is non-empty for every $\varepsilon > 0$. But fixing ε and doing all the steps of reasoning given ε essentially amounts to discretizing the model, with a loss of insight on the original model. So, I provide a two-steps argument that does not rely on a commonly known approximation of best replies, and that can be easily iterated ad infinitum.[14]

I say that a set of prices P_i^* is *unrivaled* given a set of competitor's prices P_{-i} if it contains an upper contour set of the expected profit function for each conjecture over P_{-i}. I say that a path[15] $((a_1^*, a_2^*), (p_1^*, p_2^*))$ is *compatible with forward induction reasoning* if for every unilateral deviation to $a_i \neq a_i^*$, for every set of prices P_{-i} against which the deviation is profitable,[16] for every unrivaled set P_i^* given P_{-i}, and for every unrivaled set P_{-i}^* given P_i^*, there is $\widetilde{p}_{-i} \in P_{-i}^*$ such that $\widetilde{p}_{-i} < p_{-i}$ for all $p_{-i} \in P_{-i}$. This means that the deviator cannot come to the conclusion that the competitor will certainly react with a price in a set P_{-i} of prices that make the deviation profitable, for then the competitor, anticipating this, would actually have the incentive to react with lower prices.

[14]Not assuming common knowledge of the approximation is not just cosmetic, it produces a more restrictive refinement. For not too high values of a_1, for any $\varepsilon > 0$, $p_1 = \overline{p}_2 - (a_2^* - a_1)$ is an ε-best reply to some conjecture sufficiently concentrated above $\overline{p}_2$, but given the conjecture, there is ε such that p_1 is not an ε-best reply. This will complicate the proof of Proposition 1.

[15]This notion of forward induction can be extended to *sets* of expected paths, in the same fashion as in Catonini (2020a).

[16]By a simple monotonicity argument, a deviation is profitable against every $p_{-i} \in P_{-i}$ if and only if it is profitable against every conjecture over P_{-i}, so the former formulation is chosen for simplicity.

To streamline exposition, I also suppose that firms do not deviate in case of indifference, however if firm 1 expects $\overline{p}_2$ and best replies with $p_1^F(\overline{p}_2)$, firm 2 still has the incentive to reach with.

Proposition 1 *The path* $((1/4, 3/4), (1, 1))$ *is compatible with forward induction reasoning.*

Proof. For every $a_1 \neq 1/4$, as already observed firm 1's deviation can be profitable only if at $(a_1, 3/4)$ firm 2 fixes $p_2 > \overline{p}_2$. So consider any set of prices P_2 of firm 2 with $\inf P_2 \geq \overline{p}_2$. Suppose first that $\inf P_2 \in P_2$, thus $\inf P_2 > \overline{p}_2$. Then, every unrivaled set P_i^* must contain some $p_1 \leq \inf P_2 - \Delta^a$, because every $p_1 > \inf P_2 - \Delta^a$ is an inferior reply to $\inf P_2$ with respect to some $p_1' < \inf P_2 - \Delta^a$. In turn, every unrivaled set P_2^* must contain some $p_2 < \inf P_2$, because every $p_2 \geq \inf P_2$ is an inferior reply to any $p_1 \leq \inf P_2 - \Delta^a$ with respect to some $p_2' < \inf P_2$. Suppose now that $\inf P_2 \notin P_2$. Every unrivaled set P_1^* must have $\inf P_1^* \leq \inf P_2 - \Delta^a$: if we had $\inf P_1^* > \inf P_2 - \Delta^a$, there would be $p_2 \in P_2$ with $p_2 - \Delta^a < \inf P_1^*$, and by $p_2 > \overline{p}_2$ every $p_1 \in P_1^*$ would be an inferior reply to p_2 with respect to some $p_1' < p_2 - \Delta^a$. In turn, every unrivaled set P_2^* must contain $p_2 \leq \inf P_2$, because for a conjecture on, or sufficiently concentrated above $\inf P_1^*$, every $p_2 > \inf P_2$ brings zero demand or does worse than $\inf P_2$. ■

3.2 Uniqueness

In this section I prove that $(1/4, 3/4)$ is the unique symmetric solution to the location problem, even when the focus is expanded from pure equilibria to Self-Enforcing Sets.

Theorem 2 *No symmetric location pair* $(a_1, a_2) \neq (1/4, 3/4)$ *is induced by a Self-Enforcing Set in backwards rationalizable strategies.*

It is easy to see that a Self-Enforcing set must prescribe a set of prices closed under rational behavior (CURB set) on path.[17] After deviations, backwards rationalizability only allows to fix rationalizable prices of the static pricing

[17] One might be tempted to use directly CURB sets of the strategic form in place of Self-Enforcing Sets. The problem is that no location pair is induced by a CURB set, not even when the strategic-form is reduced to the backwards rationalizable profiles, because there is no strict incentive to fix any particular rationalizable price after a deviation.

game. I am going to show that for every symmetric location pair different from $(1/4, 3/4)$ and for every subsequent CURB set of prices, the profit against the lowest price(s) of the competitor is lower than against any rationalizable price of the competitor after some unilateral deviation to a different location. Therefore, condition 1 of a Self-Enforcing Set cannot be satisfied.

Preliminarily, I show that whenever the two firms are located in different halves of the spectrum, they have no rationalizable price above p_i^*.

Claim 1: Fix (a_1, a_2) with $a_1 < 1/2 < a_2$. For each $i = 1, 2$, every $p_i > p_i^*$ is eliminated at some finite step of rationalizability.
Proof. I am going to show that, for every $k > 0$, there is $\varepsilon > 0$ such that, for every $k' > k$ and $i = 1, 2$, every $\widetilde{p}_i > p_i^* + k' - \varepsilon$ is dominated over $\left[0, p_{-i}^* + k'\right]$. Then, starting from prohibitively high prices and iteratively eliminating the top ε-interval of both firms, all prices above $p_i^* + k$ are eliminated. By $a_1 < 1/2 < a_2$, we have $\overline{\varepsilon} := \Delta^a - |p_1^* - p_2^*| > 0$. Let $\varepsilon = \min\{k/2, \overline{\varepsilon}\}$. For each $p_{-i} \leq p_{-i}^* + k'$, we have $\widetilde{p}_i > p_{-i} - \Delta^a$ and $\widetilde{p}_i > p_i^* + k'/2 = p_i^F(p_{-i}^* + k') \geq p_i^F(p_{-i})$. Hence, for every conjecture over $\left[0, p_{-i}^* + k'\right]$, $\widetilde{p}_i$ does worse than some lower price. ■

Now I show that whenever both firms are located farther from the center with respect to $(1/4, 3/4)$, also the prices below p_i^* are not rationalizable, hence the equilibrium (p_1^*, p_2^*) is the only rationalizable price pair. This implies that it is also the unique CURB set. When a firm, say firm 1, unilaterally relocates from a_1 to $a_1' \in (a_1, 1/4)$, its equilibrium revenues increase, thus the desired comparison follows.

Claim 2: Fix (a_1, a_2) with $a_1 < 1/4$ and $a_2 > 3/4$. The only rationalizable price pair is (p_1^*, p_2^*).
Proof. Without loss of generality, suppose that firm 2 is not farther from the center, thus $a_1 + a_2 \leq 1$ and $p_1^* \leq p_2^*$. First I show there is $\varepsilon > 0$ such that, for each $i = 1, 2$, every $p_i < p_2^* - \Delta^a + \varepsilon$ is eliminated at some finite step of

rationalizability. Fix $\varepsilon > 0$ such that

$$2(p_2^* - \Delta^a + \varepsilon) < p_1^* + a_1 + a_2 - 1 - \varepsilon; \tag{5}$$

to see it exists, note that for $\varepsilon = 0$ the inequality is satisfied when $a_1 + a_2 = 1$ ($p_1^* = p_2^* = 1$), and it remains satisfied when a_1 is decreased, because left- and right-hand side decrease by the same amount. Fix $k \geq \Delta^a$ and $i = 1, 2$. Fix $\widetilde{p}_i \in [p_2^* - k, p_2^* - k + \varepsilon]$. I will show $\widetilde{p}_i$ is dominated over $[p_2^* - k, \infty)$ by $2\widetilde{p}_i$; then, at least the bottom ε-interval of prices of one firm is eliminated at every step until every $p_i < p_2^* - \Delta^a + \varepsilon$ is eliminated. Fix $p_{-i} \geq p_2^* - k$. We have

$$2\widetilde{p}_i \leq 2(p_2^* - k + \varepsilon) = 2(p_2^* - \Delta^a + \varepsilon) - 2(k - \Delta^a) < p_1^* - (k - \Delta^a) \leq p_{-i} + \Delta^a,$$

$$2\widetilde{p}_i - p_i^F(p_{-i}) \leq 2\widetilde{p}_i - p_1^F(p_2^* - k) = 2\widetilde{p}_i - \left(p_1^* - \frac{k}{2}\right) < a_1 + a_2 - 1 - \varepsilon + \frac{k}{2} \leq$$

$$\leq \frac{k}{2} + \frac{2(a_1 + a_2 - 1)}{3} - \varepsilon = \frac{k}{2} + (p_1^* - p_2^*) - \varepsilon = \left(p_1^* - \frac{k}{2}\right) - (p_2^* - k + \varepsilon) \leq p_i^F(p_{-i}) - \widetilde{p}_i,$$

where the strict inequalities come from (5). Hence, $2\widetilde{p}_i$ is lower than $p_{-i} + \Delta^a$ and closer than $\widetilde{p}_i$ to $p_i^F(p_{-i})$, so it does better than $\widetilde{p}_i$ if $p_{-i} < \widetilde{p}_i + \Delta^a$. If $p_{-i} \geq \widetilde{p}_i + \Delta^a$, $\widetilde{p}_i$ brings revenues at most $\widetilde{p}_i$, while $2\widetilde{p}_i$ brings at least demand

$$\frac{a_1 + a_2}{2} + \frac{\widetilde{p}_i + \Delta^a - 2\widetilde{p}_i}{2} = a_2 - \frac{\widetilde{p}_i}{2} \geq a_2 - \frac{p_2^* - \Delta^a + \varepsilon}{2} > \frac{3}{4} - \frac{1}{4} = \frac{1}{2},$$

where the strict inequality comes from (5), $p_1^* < 1$, and $a_1 + a_2 \leq 1$.

Now fix a step m such that, for each $i = 1, 2$, every $p_i < p_2^* - \Delta^a + \varepsilon$ and, by Claim 1 and $p_2^* > p_1^*$, every $p_i > p_2^* + \varepsilon/2$ are eliminated. For each $(p_1, p_2) \in [p_2^* - \Delta^a + \varepsilon, p_2^* + \varepsilon/2]^2$, we have $|p_1 - p_2| < \Delta^a$. Then, the best reply to any conjecture over the step-m prices of the competitor with mean p_{-i} is $p_i^F(p_{-i})$. So, $(p_1^F(\cdot), p_2^F(\cdot))$ defines a contraction mapping that converges to (p_1^*, p_2^*). $\square$

Now consider a location pair (a_1, a_2) with $a_1 > 1/4$ and $a_2 = 1 - a_2$. I will drop the subscript from price thresholds that, by symmetry, are identical for both firms. First, I determine an upper bound for the lowest price of a symmetric CURB set $P \times P$.

Claim 3: There exists $p \in P$ with $p \leq \overline{p}$. Moreover, $\inf P \leq \max\left\{\overline{p} - \Delta^a, \underline{p}\right\}$.

Proof. While it is intuitive that P cannot be entirely above $\overline{p}$ given the undercutting incentives, the possible absence of best replies to an atomic conjecture makes the formal proof quite tedious, so it is deferred to the end of the section. Throughout the proof, recall that all the best replies to conjectures over P are in P by closedness under rational behavior and symmetry.[18] Taking for granted that there is $p \in P$ with $p \leq \overline{p}$, suppose by contradiction that $\inf P > \max\left\{\overline{p} - \Delta^a, \underline{p}\right\}$.

Suppose first that $\underline{p} \geq \overline{p} - \Delta^a$. By $p^F(\underline{p}) = \underline{p} + \Delta^a$ and $\inf P > \underline{p}$, we get $\overline{p} < p^F(\inf P) < \inf P + \Delta^a$. Fix $p \leq \overline{p}$ equal or close to $\inf P$ so that $p^F(p) < \inf P + \Delta^a$ as well. Perturbing the conjecture on p with small probability on $p^F(p)$, we obtain a continuum of best replies above $p^F(p) > \overline{p}$. Then, we can take a sufficiently concentrated distribution over this continuum with best reply $p^F(p) - \Delta^a < \inf P$, a contradiction.

Suppose now that $\overline{p} - \Delta^a > \underline{p}$. If $p^F(\inf P) > \overline{p}$, the argument above applies. Otherwise, start from $p \leq \overline{p}$ equal or close to $\inf P$ and iterate best replies until we find p and $p' = p^F(p)$ such that $p' \leq \overline{p} < p^F(p')$.[19] I will show there are distributions over $\{p, p'\}$ that generate a continuum of best replies between $\overline{p}$ and $\inf P + \Delta^a$. Then, we can take a sufficiently concentrated distribution over this continuum with best reply below $\inf P$, a contradiction. Fix any distribution ν over $\{p, p'\}$ such that $\widetilde{p}^* := p^F(\mathbb{E}_\nu(\cdot)) \in (\overline{p}, p + \Delta^a)$. Every $\widetilde{p} \leq p - \Delta^a$ is dominated over $\{p, p'\}$ by $p^F(p)$. Every $\widetilde{p} > p^F(p')$ is dominated over $\{p, p'\}$ by $p^F(p')$. The expected revenues from $\widetilde{p} \in \left(p' - \Delta^a, p^F(p')\right]$ are

[18] Symmetry of prices can be dispensed with, but it complicates the proofs.

[19] Recall that, as long as $\widetilde{p} \in \left(\underline{p}, \overline{p}\right]$, the best reply is $p^F(\widetilde{p})$, and the iteration of $p^F(\cdot)$ brings to $p^* > \overline{p}$.

bounded above by[20]

$$\widetilde{p}\left(\nu(p)\frac{1+p-\widetilde{p}}{2}+\nu(p')\frac{1+p'-\widetilde{p}}{2}\right)=\widetilde{p}\frac{1+\mathbb{E}_\nu(\cdot)-\widetilde{p}}{2}. \tag{6}$$

The maximum of (6) is at $\widetilde{p}=\widetilde{p}^*$ and represents the true expected revenues. There remains to show they are higher than the supremum of the expected revenues with $\widetilde{p}\in(p-\Delta^a,p'-\Delta^a]$. We have

$$\begin{aligned}
\widetilde{p}^*\left(\nu(p)\frac{1+p-\widetilde{p}^*}{2}+\nu(p')\frac{1+p'-\widetilde{p}^*}{2}\right) &> p^F(p')\left(\nu(p)\frac{1+p-p^F(p')}{2}+\nu(p')\frac{1+p'-p^F(p')}{2}\right)\\
(p'-\Delta^a)\left(\nu(p)\frac{1+p-(p'-\Delta^a)}{2}+\nu(p')\right) &> \widetilde{p}\left(\nu(p)\frac{1+p-\widetilde{p}}{2}+\nu(p')\right),
\end{aligned}$$

where the second inequality follows from the fact that $p^F(p')$ is closer than $p'-\Delta^a$ to $p^F(p)=p'$, and from $p'\leq\overline{p}$, and the third inequality from the fact that $p'-\Delta^a$ is closer than $\widetilde{p}$ to $p^F(p)=p'$. □

To vizualize, the upper bound for $\inf P$ is $\overline{p}$ for low values of a_2, $\underline{p}$ for intermediate values, and $\overline{p}-\Delta^a$ for high values.

The goal now is to determine a unilateral deviation by firm 1 to a location a_1' such that, at (a_1',a_2):

1. There is no rationalizable price of firm 2 below $p_2^L:=\min\left\{p_2^F(\overline{p}_2-(a_2-a_1')),\overline{p}_2\right\}$;
2. Firm 1's optimal profit is higher at (a_1',a_2) against p_2^L than at (a_1,a_2) against some $p\in P$

If firm 1 moves to location $a_1'=0$, firm 2 has no incentive to undercut, so it is easy to show point 1.
Claim 4: At $(0,a_2)$, $p_2^L=\min\left\{p_2^F(\overline{p}_2-a_2),\overline{p}_2\right\}$ is a lower bound of firm 2's rationalizable prices.

[20] For $\widetilde{p}\leq p^F(p')<1$, $1+p-\widetilde{p}$ is always positive, so demand is overestimated and not underestimated in case $\widetilde{p}\geq p+(a_2-a_1)$.

Proof. For every p_1, let $p_2^+(p_1) := \min\left\{p_2^F(p_1), p_1 + a_2\right\}$. Every $p_2 < p_2^+(p_1)$ is dominated over $[p_1, \infty)$ by any $p_2' \in \left(p_2, p_2^+(p_1)\right)$. I will show there is $\varepsilon > 0$ such that, for every $k \geq \varepsilon$, letting $\widetilde{p}_1 := \overline{p}_2 - a_2 - k$, every $p_1 \leq \widetilde{p}_1 + \varepsilon$ is dominated over $\left[p_2^+(\widetilde{p}_1), \infty\right)$. So, we can iteratively eliminate firm 2's dominated prices and firm 1's ε-bottom of prices, until every $p_1 \leq \overline{p}_2 - a_2$ and every $p_2 < p_2^L$ are eliminated (in finitely many steps).

In the next paragraph, I will show there is $\widetilde{p}_1^* < \overline{p}_2 - a_2$ such that $p_1 = \overline{p}_2 - a_2$ is dominated over $\left[p_2^+(\widetilde{p}_1^*), \infty\right)$ by $p_1' = p_1^F(\overline{p}_2)$. Then, let $\varepsilon := \overline{p}_2 - a_2 - \widetilde{p}_1^*$. For every $k \geq \varepsilon$ and $j \geq k - \varepsilon$, $p_1 = \overline{p}_2 - a_2 - j$ is dominated over $P_2 := \left[p_2^+(\widetilde{p}_1^*) - j, \infty\right)$ by $p_1' = p_1^F(\overline{p}_2) - j$, because reducing both firms' prices by j keeps demands constant and thus reduces revenues more when demand is higher, i.e., more under p_1 than under p_1'. Moreover, letting $\widetilde{p}_1 := \overline{p}_2 - a_2 - k$, we have $P_2 \supseteq \left[p_2^+(\widetilde{p}_1), \infty\right)$, because

$$\begin{aligned} p_2^F(\widetilde{p}_1^*) - j &\leq p_2^F(\widetilde{p}_1^*) - k + \varepsilon \leq p_2^F(\widetilde{p}_1^* - k + \varepsilon) = p_2^F(\widetilde{p}_1), \\ \widetilde{p}_1^* + a_2 - j &\leq \widetilde{p}_1^* + a_2 - k + \varepsilon = \widetilde{p}_1 + a_2. \end{aligned}$$

By $\underline{p}_2 < 0$, $p_1^F(\overline{p}_2)$ is the (only) best reply to $\overline{p}_2$ and brings profit $\overline{p}_2 - a_2$, so it brings higher profit than $p_1 = \overline{p}_2 - a_2$ also against every $p_2 > \overline{p}_2$. Since firm 2 is closer to the center,

$$p_1^F(\overline{p}_2) < p_2^F(\overline{p}_2) < p_2^F(\overline{p}_2 - a_2) + a_2. \tag{7}$$

Moreover, $p_1^F(\overline{p}_2)$ is closer than p_1 to $p_1^F(p_2^F(\overline{p}_2 - a_2))$: the inequality

$$p_1^F(\overline{p}_2) - p_1^F(p_2^F(\overline{p}_2 - a_2)) < p_1^F(p_2^F(\overline{p}_2 - a_2)) - (\overline{p}_2 - a_2) \tag{8}$$

reduces to $\overline{p}_2 < 1 + a_2/2$, which is true for $a_2 = 3/4$ and thus also for lower values of a_2. Fix $\widetilde{p}_1^* < \overline{p}_2 - a_2$ such that (7) and (8) are preserved with $p_2^F(\widetilde{p}_1^*)$ in place of $p_2^F(\overline{p}_2 - a_2)$, and $p_1^F(\overline{p}_2)$ is still closer than p_1 also to $\widetilde{p}_1^* + a_2$. Then, $p_1^F(\overline{p}_2)$ does better than p_1 also against each $p_2 \in \left[p_2^+(\widetilde{p}_1^*), \overline{p}_2\right)$. $\square$

For most values of a_2, $a_1' = 0$ satisfies also point 2, but not for all. A sufficient condition is $p_2^L = \overline{p}_2$, because the supremum of firm 1's profit against $\overline{p}_2$ does not depend on firm 1's location. This happens when firm 2 is sufficiently close to the center. Let $\overline{a}_2 = 4\sqrt{7} - 10 > 0.58$.

Claim 5: When $a_2 \in (1/2, \overline{a}_2)$, firm 1's profit at $(0, a_2)$ against p_2^L is higher than at (a_1, a_2) against some $p \in P$.

Proof: At $(0, a_2)$, we have $\underline{p}_1 > \overline{p}_2 - a_2$,[21] therefore $p_2^L = \overline{p}_2$. By $\underline{p}_2 < 0$, firm 1's the best reply to $\overline{p}_2$ is $p_1^F(\overline{p}_2)$, with profit $\overline{p}_2 - a_2 = 4 - 2a_2 - 4\sqrt{1 - a_2}$. At (a_1, a_2), the supremum of firm 1's profit against $\overline{p}$ is the same and is not attained when $\underline{p} \geq \overline{p}$, because then $p^F(\overline{p})$ is not a best reply. By Claim 3, there is $p \in P$ with $p \leq \overline{p}$, and $p < \overline{p}$ if $\underline{p} < \overline{p}$. □

Another sufficient condition is $\inf P \leq \overline{p} - \Delta^a$, because firm 1 always gets a higher profit at $(0, a_2)$ against p_2^L than at (a_1, a_2) against $p_2 = \overline{p} - \Delta^a$. This happens when firm 2 is sufficiently far from the center, and it is useful to observe it is maintained against $\underline{p}_2$ in place of $\overline{p} - \Delta^a$ down to $\overline{\overline{a}}_2 := \left(6 + 4\sqrt{39}\right)/49 < 0.64$.

Claim 6: When $a_2 \in \left(\overline{\overline{a}}_2, 3/4\right)$, firm 1's profit at $(0, a_2)$ against p_2^L is higher than at (a_1, a_2) against some $p \in P$.

Proof: At $(0, a_2)$, by $\underline{p}_2 < 0$ and $p_2^L = p_2^F(\overline{p}_2 - a_2) < \overline{p}_2$ (the case $p_2^L = \overline{p}_2$ is exhausted by Claim 5), firm 1's best reply to p_2^L is $p_1^F(p_2^L)$, and is larger than $p^F(\overline{p} - \Delta^a)$ and $p^F(\underline{p})$ at (a_1, a_2).[22] So firm 1's profit is exactly $\left(p_1^F(p_2^L)\right)^2/2$ at $(0, a_2)$ against p_2^L, while at (a_1, a_2) recall that the profit is bounded above by $\left(p^F(p)\right)^2/2$ against any $p \leq \overline{p}$. By Claim 3, $\inf P \leq \overline{p}$, and also $\inf P \leq \max\left\{\overline{p} - \Delta^a, \underline{p}\right\}$ (here actually $\underline{p} < \overline{p}$). Thus, by continuity of $p^F(\cdot)$, there is $p \in P$ such that the desired comparison of profits holds through. □

Finally, for intermediate values of a_2, moving to location 0 does not guarantee to firm 1 enough profits. However, moving to $a_1' = 0.15$ will do, while preserving the lower bound of firm 2's rationalizable prices. This completes the proof of Theorem 2. Let $\Delta' := a_2 - a_1'$.

[21] The inequality reads $2 - 3a_2 > 4 - 2a_2 - 4\sqrt{1 - a_2}$, and it is an equality for $a_2 = \overline{a}_2$.

[22] The first comparison reads $\frac{3}{2} - \sqrt{1 - a_2} - \frac{a_2}{4} > \frac{5}{2} - a_2 - 2\sqrt{1 - a_2}$ and holds for all $a_2 \in \left(\frac{1}{2}, \frac{3}{4}\right)$, while the comparison with $p^F(\underline{p}) = 2 - 2a_2$ yields the threshold $\overline{\overline{a}}_2$.

Claim 7: When $a_2 \in \left[\overline{a}_2, \overline{\overline{a}}_2\right]$, firm 1's profit at (a_1', a_2) against p_2^L is higher than at (a_1, a_2) against some $p \in P$. Moreover, at (a_1', a_2), there is no rationalizable price of firm 2 above p_2^L.

Proof. Increasing a_2 from $\overline{a}_2$, we have $p_2^L = \overline{p}_2$ at (a_1', a_2) as long as $\overline{p} \leq \underline{p}$ at (a_1, a_2),[23] and the argument of Claim 5 applies. Then, since $p^F(\underline{p})$ decreases and $p_1^F(p_2^L)$ increases, the argument of Claim 6 applies up to $\overline{\overline{a}}_2$.(where $\underline{p}$ is still higher than $p - \Delta^a$).

For rationalizability, we can first eliminate all prices above $\widetilde{p}_1^* = 0.93 > p_1^*$ by Claim 1, and then all the prices above p_2^L can be easily eliminated in only three steps. Recall that $\left[\overline{a}_2, \overline{\overline{a}}_2\right] \subset [0.58, 0.64]$, thus $\Delta' > 0.43$.

1. Every $p_2 < 0.34$ is dominated by $p_2' = 5p_2/4 < \Delta'$: for $p_1 \in [0, p_2 + \Delta')$, by $p_2^F(p_1) \geq p_2^F(0) > p_2'$ firm 2's profit is strictly increasing over $[0, p_2']$; for $p_1 \geq p_2 + \Delta'$, firm 2's demand under p_2' is higher than $4/5$,[24] thus the profit is higher than p_2.

2. Every $p_1 < \overline{p}_2 - \Delta' < 0.32$ ([25]) is dominated over $[0.34, \infty)$ by $p_1' := p_1^F(p_1 + \Delta')$: p_1' is the unique best reply to $p_2 = p_1 + \Delta' < \overline{p}_2$ ($\underline{p}_2 < 0$), and for $p_2 > p_1 + \Delta'$ firm 1's profit with p_1' weakly increases, while with p_1 it remains p_1; for $p_2 \in [0.34, p_1 + \Delta')$, $p_1' = a_2 + p_1/2 < p_2 + \Delta'$ is closer than p_1 to $p_1^F(p_2) = (a_1' + a_2 + p_2)/2 > a_2/2 + 3p_1/4$.

3. Every $p_2 < p^L$ is dominated over $[\overline{p}_2 - \Delta', \widetilde{p}_1^*]$. Each $p_2 \in \left(\widetilde{p}_1^* - \Delta', p^L\right)$ is dominated by any $p_2' \in (p_2, p^L)$, because for each $p_1 \in [\overline{p}_2 - \Delta', \widetilde{p}_1^*]$ firm 2's profit is strictly increasing over $\left[p_2, p^L\right)$. Each $p_2 \leq \widetilde{p}_1^* - \Delta' < 1/2$ is dominated by $p_2' := 4p_2/3$: for each $p_1 \in [\overline{p}_2 - \Delta', p_2 + \Delta')$, firm 2's profit is strictly increasing over $[p_2, p_2']$ by $p_2' < \min\left\{p_1 + \Delta', p_2^F(p_1)\right\}$;[26]

[23] For $a_2 \approx 0.62$ that solves $\overline{p} = \underline{p}$ at (a_1, a_2), i.e. $3 - 4a_2 = 3 - 4\sqrt{1 - a_2}$, we get $\underline{p}_1 > \overline{p}_2 - \Delta'$ at (a_1', a_2), i.e. $2 - 3a_2 + a_1' > 4 - 2a_2 - 4\sqrt{1 - a_2}$. This inequality is preserved for lower values of a_2.

[24] Under p_2 all customers weakly prefer to buy from firm 2, therefore under p_2' not more than the leftmost $a_1' + (p_2' - p_2)/2 < 0.2$ customers are lost.

[25] 0.32 is the value of $\overline{p}_2 - \Delta'$ for $a_2 = 0.64$.

[26] We have $p_2' < 2/3$, while $p_1 + \Delta' \geq \overline{p}_2 > 2/3$ and $p_2^F(p_1) \geq p_2^F(\overline{p}_2 - \Delta') > 2/3$ as well.

for each $p_1 \in [p_2 + \Delta', \widetilde{p}_1^*]$, firm 2's demand under p_2' is higher than $3/4$.[27]

Proof of Claim 3 (first part). I show that every CURB set has a price not larger than $\overline{p}$. I do so by showing that every best response set entirely above $\overline{p}$ is not closed under rational behavior. I maintain the symmetry assumption to simplify exposition, although it is never used. So, let $P \times P$ be a best response set with $p > \overline{p}$ for every $p \in P$. Let $p^I := \inf P$. By Claim 1, the prices above p^* cannot be in P because they are not rationalizable. Moreover, (p^*, p^*) is not an equilibrium, thus not a best response set, and for every $p < p^*$ we have $p^F(p) > p$. Then, P must contain at least one sequence $(p^k)_{k>0}$ that converges to $p^I + \Delta^a$ from above, otherwise p^I or a price just above would not be justified. This sequence supports a conjecture that justifies p^I, so if $p^I \notin P$, then P is not closed under rational behavior. If $p^I \in P$, then by assumption $p^I > \overline{p}$. I am going to show the exixtence of either a sequence in P that converges to some $p < p^I + \Delta^a$ from above, which supports a conjecture that justifies $p - \Delta^a < p^I$, or an interval of best replies to conjectures over P, which, iterating, generates best replies below p^I as well.

Note first that $\sup P \leq 4\Delta^a$, because every $p > 4\Delta^a$ is dominated over $[0, p]$ by any $p' \in (p/2, p - 2\Delta^a)$: whenever p gives positive demand, it is at most $1/2$, and p' gives demand 1. Moreover, for each $k > 0$, we have $p^k > \overline{p} + \Delta^a > 3\Delta^a$,[28] so p^k does not undercut any price in P. Without loss of generality, we can take a conjecture $\nu^k \in \Delta(P)$ that justifies p^k with $\nu^k\big((0, p^k - \Delta^a)\big) = 0$. Now, either P contains a sequence that converges to p^I from above, or $P \cap (p^I, p^I + \varepsilon) = \emptyset$ for some $\varepsilon > 0$. In this second case, there is $\widetilde{k} > 0$ such that, for every $k > \widetilde{k}$, $\nu^k\big(\big[0, p^k - \Delta^a + \delta\big]\big) = 0$ for some $\delta > 0$. Then, to justify p^k, ν^k must have mean $(p^F)^{-1}(p^k)$, so p^k brings expected profit $\left(p^k\right)^2/2$.

If for every k there is $l > k$ such that $\inf(\mathrm{Supp}\nu^l) < \inf(\mathrm{Supp}\nu^k)$, there is a sequence in P that converges to some $p \leq \left(p^F\right)^{-1}(p^I + \Delta^a) < p^I +$

[27] Under p_2 all customers weakly prefer to buy from firm 2, therefore under p_2' not more than the leftmost $a_1' + (p_2' - p_2)/2 < 0.25$ customers are lost ($p_2' - p_2 = p_2/3 < 1/6$).

[28] By $a_1 + a_2 = 1$ and $a_1 > 1/4$, $\overline{p} = 3 - 4\sqrt{a_1} > 2 - 4a_1 = 2\Delta^a$ (the inequality is an equality for $a_1 = 1/4$).

Δ^a. So suppose there is $\overline{k} > \widetilde{k}$ such that, for every $k > \overline{k}$, $\inf(\mathrm{Supp}\nu^k) \geq \inf(\mathrm{Supp}\nu^{\overline{k}}) =: p^S$. If there is $k > \overline{k}$ with $p^S < \inf(\mathrm{Supp}\nu^k) =: \widetilde{p}^S$, by $\widetilde{p}^S - \Delta^a \leq \left(p^k\right)^2/2 < (p^{\overline{k}})^2/2$, we can modify $\nu^{\overline{k}}$ by moving probability from below $\widetilde{p}^S$ to $\widetilde{p}^S$ or just above, and we obtain a continuum of best replies above $p^{\overline{k}}$. If $\inf(\mathrm{Supp}\nu^k) = p^S$ for every $k > \overline{k}$, fix $l > k > j > \overline{k}$. Let ν be the convex combination of ν^j and ν^l with weights α and $1-\alpha$ so that the mean of ν is $(p^F)^{-1}(p^k)$. Since ν and ν^k have the same mean, one cannot first-order stochastically dominate the other, so we have

$$p' := \sup\left\{p \in P | \nu^k([p,\infty)) \geq \nu([p,\infty))\right\} > p^S.$$

Create a new label $\widetilde{p}'$ for p', let the intervals $[p',\infty)$ and $[0,\widetilde{p}']$ include only the indicated label, and redistribute $\nu^k(p')$ between p' and $\widetilde{p}'$ to obtain $\widetilde{\nu}^k$ with $\nu([p',\infty)) = \widetilde{\nu}^k([p',\infty)) =: \beta$ and $\nu([0,p')) = \widetilde{\nu}^k([0,\widetilde{p}'))$.[29] For both ν and $\widetilde{\nu}^k$, the common mean can be written as the convex combination with weights β and $1-\beta$ of the means conditional on $[p',\infty)$ and $[0,\widetilde{p}']$. Since the mean conditional on $[p',\infty)$ is not larger for ν^k than for ν, the opposite holds for the mean conditional on $[0,\widetilde{p}']$, and $\widetilde{\nu}^k(\widetilde{p}') \geq \nu(\widetilde{p}') = 0$. This means that, for some $\varepsilon > 0$, every $p \in [p' - \varepsilon - \Delta^a, p' - \Delta^a]$ gives an expected payoff not higher than $\left(p^k\right)^2/2$ not only under ν^k but also under ν. So, by $\left(p^k\right)^2 < \alpha\left(p^j\right)^2 + (1-\alpha)\left(p^l\right)^2$, there is $\delta > 0$ such that (i) the expected profit from every $p \in [p' - \delta - \Delta^a, p' - \Delta^a]$ is bounded away from the optimum either under ν^j or under ν^l, say ν^j, and (ii) $\nu^j([p'-\delta, p')) > 0$.[30] Then, we can perturb ν^j by moving small probability from $[p'-\delta, p')$ to p' or to a price just below, so that the prices in $[p' - \delta - \Delta^a, p' - \Delta^a]$ remain suboptimal. The expected demand is unchanged for the prices below $p' - \delta - \Delta^a$, while for all prices above

[29]Note that if $\nu^k(p') = 0$, then we already have the two desired equalities. Otherwise, the exact modification is the following: $\widetilde{\nu}^k(\widetilde{p}') = \nu^k([p',\infty)) - \nu([p',\infty))$ and $\widetilde{\nu}^k(p') = \nu^k(p') - \widetilde{\nu}^k(\widetilde{p}')$. It is well defined because by definition of p', $0 \leq \nu^k([p',\infty)) - \nu([p',\infty)) \leq \nu^k(p')$.

[30]Requirement (ii) is compatible with (i) because as long as there is zero density below p', below $p' - \Delta^a$ a decrease in price entails a decrease of expected revenues. This is because the probability of undercutting does not increase and is already higher than at the optimal price (while the probability of being undercut is always zero), therefore at the optimal price the same marginal decrease in price (which obviously does not increase expected revenues) entails a higher increase of expected demand, and insists on a lower expected demand.

$p' - \Delta^a$ the increase of expected demand is the same. So, the best reply to the perturbed ν^j with mean p is $p^F(p)$, and we generate a continuum of best replies just above p^j. ■

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