Утв1. Для оценок МНК коэффициентов β_0 и β_1 уравнения регрессии $Y_i = \beta_0 + \beta_1 X_i + \varepsilon_i$ при выполнении условий ТГМ:

$$\operatorname{var}(\hat\beta_1) = \frac{\sigma_\varepsilon^2}{\sum_{i=1}^n x_i^2}; \quad \operatorname{var}(\hat\beta_0) = \sigma_\varepsilon^2 \frac{\sum_{i=1}^n X_i^2}{n \sum_{i=1}^n x_i^2} = \sigma_\varepsilon^2\left(\frac{1}{n} + \frac{\bar X^2}{\sum_{i=1}^n x_i^2}\right)$$

$$\operatorname{cov}(\hat\beta_0; \hat\beta_1) = -\frac{\bar X \cdot \sigma_\varepsilon^2}{\sum_{i=1}^n x_i^2}, \text{ где } x_i = X_i - \bar X, \; y_i = Y_i - \bar Y$$

$$\Delta \quad \hat\beta_1 = \frac{\sum_{i=1}^n x_i y_i}{\sum_{j=1}^n x_j^2} = \sum_{i=1}^n \frac{x_i}{\sum_{j=1}^n x_j^2} y_i = \sum_{i=1}^n \omega_i y_i = \sum_{i=1}^n \omega_i (Y_i - \bar Y) =$$

$$= \sum_{i=1}^n \omega_i Y_i - \bar Y \underbrace{\sum_{i=1}^n \omega_i}_{0} = \sum_{i=1}^n \omega_i (\beta_0 + \beta_1 X_i + \varepsilon_i) = \beta_0 \underbrace{\sum_{i=1}^n \omega_i}_{0} +$$

$$+ \beta_1 \sum_{i=1}^n \omega_i \underbrace{X_i}_{x_i + \bar X} + \sum_{i=1}^n \omega_i \varepsilon_i = \beta_1 \Big(\underbrace{\sum_{i=1}^n \omega_i x_i}_{1} + \bar X \underbrace{\sum_{i=1}^n \omega_i}_{0} \Big) + \sum_{i=1}^n \omega_i \varepsilon_i$$

$$\operatorname{var}(\hat\beta_1) = \sum_{i=1}^n \omega_i^2 \cdot \sigma_\varepsilon^2 = \frac{\sigma_\varepsilon^2}{\sum_{i=1}^n x_i^2}$$

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$$\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{X} = \frac{1}{n}\sum_{i=1}^{n} Y_i - \sum_{i=1}^{n} \omega_i (Y_i - \bar{Y}) \cdot \bar{X} = \sum_{i=1}^{n} \left(\frac{1}{n} - \omega_i \bar{X}\right) Y_i + \bar{X}\bar{Y} \underbrace{\ldots}_{0}$$

$$= \sum_{i=1}^{n} \left(\frac{1}{n} - \omega_i \bar{X}\right) \cdot Y_i \Rightarrow$$

$$var(\hat{\beta}_0) = \sum_{i=1}^{n} \left(\frac{1}{n^2} - 2 \cdot \frac{1}{n}\omega_i \bar{X} + \omega_i^2 \bar{X}^2\right) \cdot \sigma_\varepsilon^2 =$$

$$= \sigma_\varepsilon^2 \left(\frac{1}{n} - \frac{2}{n} \bar{X} \underbrace{\sum_{i=1}^{n} \omega_i}_{0} + \bar{X}^2 \underbrace{\sum_{i=1}^{n} \omega_i^2}_{= \frac{1}{\sum_{i=1}^{n} x_i^2}}\right) = \sigma_\varepsilon^2 \left(\frac{1}{n} + \frac{\bar{X}^2}{\sum_{i=1}^{n} x_i^2}\right) =$$

$$= \sigma_\varepsilon^2 \frac{\sum_{i=1}^{n} x_i^2 + n\bar{X}^2}{n \cdot \sum_{i=1}^{n} x_i^2} = \sigma_\varepsilon^2 \frac{\sum X_i^2 - n\bar{X}^2 + n\bar{X}^2}{n \sum_{i=1}^{n} x_i^2} = \sigma_\varepsilon^2 \frac{\sum X_i^2}{n \sum_{i=1}^{n} x_i^2}$$

$cov(\hat{\beta}_0, \hat{\beta}_1)$ — ?

$\bar{Y} = \hat{\beta}_0 + \hat{\beta}_1 \bar{X}$, $var(\bar{Y}) = \frac{1}{n}\sigma_\varepsilon^2$

$$var(\hat{\beta}_0 + \hat{\beta}_1 \bar{X}) = var(\hat{\beta}_0) + 2\bar{X} cov(\hat{\beta}_0, \hat{\beta}_1) + \bar{X}^2 var(\hat{\beta}_1)$$

$$\frac{1}{n}\sigma_\varepsilon^2 = \sigma_\varepsilon^2 \left(\frac{1}{n} + \frac{\bar{X}^2}{\sum_{i=1}^{n} x_i^2}\right) + 2\bar{X} cov(\hat{\beta}_0, \hat{\beta}_1) + \bar{X}^2 \cdot \frac{\sigma_\varepsilon^2}{\sum_{i=1}^{n} x_i^2} \Rightarrow$$

$$cov(\hat{\beta}_0, \hat{\beta}_1) = -\frac{\bar{X} \cdot \sigma_\varepsilon^2}{\sum_{i=1}^{n} x_i^2}$$

Утв. 2 При выполнении условий ТГМ оценки $\hat{\beta}_0^{МНК}$ и $\hat{\beta}_1^{МНК}$ являются BEST, т.е. имеют наименьшую дисперсию в классе всех линейных несмещенных оценок.

$\triangleright$ Пусть $\tilde{\beta}_1 = \sum_{i=1}^{n} \tilde{\omega}_i Y_i$ — другая несмещенная оценка, т.е. $E(\tilde{\beta}_1) = \beta_1 \Rightarrow E(\tilde{\beta}_1) = \sum_{i=1}^{n} \tilde{\omega}_i E(Y_i) =$

$= \sum_{i=1}^{n} \tilde{\omega}_i E(\beta_0 + \beta_1 X_i + \varepsilon_i) = \beta_0 \sum_{i=1}^{n} \tilde{\omega}_i + \beta_1 \sum_{i=1}^{n} \tilde{\omega}_i X_i = \beta_1 \Rightarrow$

(1) $\sum_{i=1}^{n} \tilde{\omega}_i = 0$; (2) $\sum_{i=1}^{n} \tilde{\omega}_i X_i = 1$, т.е. необходимо решить задачу:

$\mathrm{var}(\tilde{\beta}_1) = \sigma_\varepsilon^2 \sum_{i=1}^{n} \tilde{\omega}_i^2 \to \min$

при ограничениях (1) и (2)

$$\sum_{i=1}^{n} \tilde{\omega}_i^2 = \sum_{i=1}^{n} (\tilde{\omega}_i - \omega_i + \omega_i)^2 = \sum_{i=1}^{n} (\tilde{\omega}_i - \omega_i)^2 + 2\sum_{i=1}^{n} (\tilde{\omega}_i - \omega_i)\omega_i + \sum_{i=1}^{n} \omega_i^2$$

$$\sum_{i=1}^{n} (\tilde{\omega}_i - \omega_i)\omega_i = \sum_{i=1}^{n} \tilde{\omega}_i \omega_i - \sum_{i=1}^{n} \omega_i^2 = \sum_{i=1}^{n} \tilde{\omega}_i \frac{(X_i - \bar{X})}{\sum_{j=1}^{n} x_j^2} - \sum_{i=1}^{n} \omega_i^2 =$$

$$= \frac{1}{\sum_{j=1}^{n} x_j^2} \Big(\underbrace{\sum_{i=1}^{n} \tilde{\omega}_i X_i}_{1} - \bar{X} \underbrace{\sum_{i=1}^{n} \tilde{\omega}_i}_{0} \Big) - \frac{1}{\sum_{j=1}^{n} x_j^2} = \frac{1}{\sum_{j=1}^{n} x_j^2} - \frac{1}{\sum_{j=1}^{n} x_j^2} = 0$$

$$\Rightarrow \sum_{i=1}^{n} \tilde{\omega}_i^2 = \sum_{i=1}^{n} (\tilde{\omega}_i - \omega_i)^2 + \frac{1}{\sum_{j=1}^{n} x_j^2}$$

$\Rightarrow \sum_{i=1}^{n} \tilde{\omega}_i^2$ достигает минимума при $\tilde{\omega}_i = \omega_i$, т.е. для оценок МНК. $\square$

Для $\hat{\beta}_0$ д-во аналогично.