Optimal Information Disclosure in Contests with Stochastic Prize Valuations

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Abstract

We study optimal information design in static contests where contestants do not know their values of winning. The designer aims at maximizing the total expected effort. Before the contest begins, sheы commits to the information technology that includes (1) a signal distribution conditional on each values profile (state) and (2) the type of signal disclosure to contestants – public, private or none at all. Upon observing the signal, contestants simultaneously choose effort that maximizes their expected payoff in an all-pay auction game. Using a mixture of analytical and numerical methods, we find that the optimal information technology involves private and positively correlated signals that never reveal the true state precisely if the contestants' values of winning are different. In settings where public disclosure is a prerequisite, the optimal signal distribution generates symmetric beliefs about the values profile, so that a complete information concealment is optimal, while public and precise disclosure of each state is not.

KEY WORDS: contest, all-pay auction, information disclosure, signal distribution, signal precision
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1 Introduction

Many real life competitions can be viewed as contests in which the contestants' values of winning/prize valuations are not known neither to the contest organizer (designer), nor to the contestants themselves. For example, in R&D and patent races, the designer and the competing researchers/labs/firms are often not sure about the rents they would extract from winning the patent and obtaining monopoly rights for selling their innovation. In competition for promotion among employees, there is always doubt about the goodness of match between each candidate and the new position. Similarly, a competitive selection of students for a place in a prestigious graduate program and assessment centers for some types of employment (often in management or military command)[1] involve a fair share of uncertainty about the candidates' fitness for the vacancy and the associated value of winning.

This information about the contestants' values of winning, however, is an important determinant of how much effort the contestants will be willing to exert in the competition, and that, in turn, is frequently what the designer cares about the most. For example, if the contestants know that they all have a high value of winning, they are likely to compete aggressively and exert much effort, while the opposite might be the case if the competition is known to be "uneven". Then, it is reasonable to presume that the designer, who often has an advantage in acquiring the relevant information (in the form of a report, expert evaluation, etc.),[2] may wonder whether she can benefit from (a) soliciting an informative signal about the contestants' values of winning and (b) passing this signal on to the contestants. Even if this signal is of no intrinsic interest to the designer and is meant to be used exclusively for stimulating the competitive efforts, the designer may be concerned with how precise she wants the signal to be and in what way, if any, she wants to convey it to the contestants. This is the essence of the research question that we address in this paper: can the organizer of a contest with unknown prize valuations increase the total competitive effort by choosing optimally the precision of the signal and the way of communicating it to the contestants?

[1] About 68% of employers in the UK and USA now use some form of assessment centre as part of their recruitment/promotion process (see $https://www.assessmentcentrehq.com/$).

[2] For example, she has access to application materials of all competitors, and she can invite an expert/committee or solicit a report to evaluate the unknown characteristics of both, the prize (job, patent, position in a graduate program) and the contestants themselves.

In general, players in a contest can be informed publicly or privately or not informed at all. We consider these three information regimes and assume that for each regime the designer chooses the precision of the signal and whether to make it dependent on the true (unknown) values profile. For example, in case of public information disclosure, the designer can choose to have a perfectly precise signal about any values profile, or a signal that is perfectly precise only when both contestants have high values of winning, or any other signal with predetermined precision. This provides the designer with a very broad set of possible disclosure rules.

We formalize these ideas in a stylized all-pay auction model with two ex ante identical players who compete for a single prize, and a designer who aims at maximizing the expected aggregate effort. The value of a prize can be either high or low for each contestant and is determined randomly according to a symmetric prior distribution. The designer has access to a certain information technology that allows her to (costlessly) draw signals about the contestants' prize values. She can choose (1) the precision of this information technology, conditional on each actual values profile (state of the world) and (2) whether to pass the signals on to both contestants privately, or publicly, or keep them uninformed. In the spirit of the Bayesian persuasion literature, we assume that the designer commits to the disclosure regime before she receives the signal. Moreover, irrespective of the chosen regime, she always reveals the obtained signal truthfully. After that the contestants update their beliefs about own and the opponent's type and choose an action from continuous effort space.

To infer the optimal choice of information technology by the designer, we first study equilibria of the contest game and find the best signal distribution (i.e., the signal precision parameters) for each type of information disclosure separately. Then we compare the resulting values of the expected aggregate effort across all the cases and characterize the information technology that constitutes the global optimum. Our main result is that the optimal disclosure policy employs private signals, and these signals are (slightly) positively correlated. We also find, using a combination of analytical and numerical approaches, that one specific feature of these signals is that they reveal the true state precisely if and only if both contestants have the same values of winning, that is, the state is symmetric.[3] The

[3]A deviation from such policy (towards signals that are not perfectly precise in symmetric states) can be profitable only when a gap between the high and low values of winning is sufficiently large and the prior probability of having a high value is low.

intuition for this result follows from an observation that the perceived asymmetry in contestants' values of winning reduces efforts. In that sense, the policy that reveals the information about values of winning privately and does so using signals that (a) are positively correlated and (b) never reveal the true state precisely if the contestants' values of winning are different, is best at masking out the asymmetry. The finding that private disclosure is optimal has been reported in other studies, too (e.g., Melo-Ponce, 2020), but not in the setting where contestants are a priori identical and have no private information. The latter produces a different information structure in our model, as it means that it is the designer who decides whether to provide the contestants with private information and what kind of information it should be. As a result (and in contrast to Melo-Ponce, 2020), we find that disclosure rules inducing asymmetric beliefs are never optimal, and the designer extracts the full surplus.

We further find that while public information disclosure is never optimal, in cases when it *has* to be used, an optimal signal distribution is such that it generates symmetric beliefs in contestants. That is, upon receiving a public signal from that distribution, the contestants must believe that each of them is equally likely to have high value of winning. This is, in fact, the only condition that an optimal public disclosure rule must fulfill. For example, complete nondisclosure of information, which can be regarded as public disclosure with a completely uninformative signal, satisfies this property. On the other hand, a precise public revelation of *every* state is never optimal. The key intuition behind these findings is that when disclosure induces symmetric beliefs about valuations, it makes the contestants perceive their incentives and chances of winning as equal, which results in the highest competitive efforts.

Despite the simple intuition behind the results, our paper is the first to provide a precise theoretical characterization of optimal disclosure policies in a setting with (i) a broad range of available disclosure rules, (ii) both contestants being initially uninformed about their values of winning and (iii) effort choices being continuous. In the next section we discuss our contribution to the existing literature in more detail. The rest of the paper is organized as follows. In Section 3 we provide a simple example and give a preview of our main results. In Section 4 we introduce the model. Section 5 describes equilibria of the contest game under public and private information disclosure. Section 6 characterizes the optimal information disclosure policy. Section 7 concludes.

2 Related literature

This paper lies at the intersection of two research fields. First, it is related to the Bayesian persuasion literature initiated by Kamenika and Gentzkow (2011) and extended to a setting with multiple signal receivers by a stream of subsequent studies (Alonso and Camara, 2016 a,b; Bardhi and Guo, 2018; Arieli and Babichenko, 2019; Chan et al., 2019). In this literature, the closest to us is Taneva (2018). She uses the concept of Bayes correlated equilibrium introduced by Bergemann and Morris (2016) to solve for the optimal signal structure in an environment with multiple interacting receivers. We borrow Taneva's signal parametrization but allow the precision parameters to be state-dependent. This is important in our setting since in a contest where state captures the players' values of winning, it might be optimal to make the signals more or less precise depending on the state. Our key results confirm this intuition, and perfect precision appears optimal only in those states where the contestants' values of winning are the same. Another important difference from Taneva (2018) is that our model assumes continuous efforts, while her results are derived for a discrete and finite action space, which makes them inapplicable in our setting.

Second, our study contributes to the literature on the optimal information revelation/feedback in contests. Most papers in this field focus on very simple disclosure rules and/or assume that at least one of the contestants is privately informed about own type. Aoyagi (2010) investigates the optimal feedback in two-stage tournaments and looks only at no feedback and full feedback cases. He finds that the no-feedback policy maximizes the players' expected effort when the second stage effort cost is convex; otherwise, full feedback is best.[4] Zhang and Zhou (2016) model a static two-player contest where one of the players has private information about his prize valuation. The authors show that if this valuation follows a binary distribution, one can reduce the set of information revelation policies to full disclosure and no disclosure without loss of generality. Warneryd (2003) analyzes a game between two ex-ante symmetric contestants who compete for a prize of a common but uncertain value. The disclosure regimes he studies are (1) concealment, (2) full information revelation, and (3) player-specific disclosure, where only one contestant

[4]Lai and Matros (2007), Gershkow and Perry (2009), Ederer (2010), Goltsman and Mukherjee (2011) and Mihm and Schlapp (2019) work on similar issues and also look at the restricted set of information disclosure policies.

gets informed about the prize value. Warneryd (2003) finds that the last policy results in the smallest aggregate effort. This is consistent with our finding that the best outcome is obtained when contestants have symmetric beliefs.

Lu et al. (2018) also focus on two-player all-pay auctions with incomplete information. In their setting, the contestants have binary types – either high or low – that the designer can observe. The set of disclosure policies is restricted to the following *anonymous* rules:[5] (1) full disclosure, (2) no disclosure, and (3) state-contingent disclosure when the designer commits to reveal only the "high-high" or "low-low" profile. Lu et al. (2018) rank these policies according to different criteria. In terms of the expected aggregate effort, concealment outperforms the other policies. Serena (2018) investigates almost the same set of disclosure rules in a Tullock contest with two players that are privately informed about their skills. He finds that revealing the "high-high" profile and concealing other states maximizes the aggregate effort. These results contrast with our findings for a contest where players have no private knowledge about their values. In our model, a rule where both "high-high" and "low-low" profiles are revealed privately and precisely leads to a greater expected aggregate effort than a policy that reports only the "high-high" profile.

Similarly to us, Celik and Michelucci (2020) assume no private information on the contestants' side and allow for a rich set of disclosure rules, where a principal chooses the optimal coarseness of information that she provides to the contestants. They report that the solution varies with the size of the effort cost and with the relative likelihood of different states. In contrast to us, Celik and Michelucci (2020) consider only public information disclosure, assume binary efforts and analyze two alternative settings: where the principal commits to a disclosure policy before learning the true state, and where such commitment is missing. We focus on the environment with commitment, but by comparing optimal public and private disclosure regimes, we find that the optimal private disclosure delivers a higher total effort.

Finally, the closest to our paper are Melo-Ponce (2020) and Kuang et al. (2019). Melo-Ponce (2020) uses the concept of Bayes correlated equilibrium introduced by Bergemann and Morris (2016) to find the optimal disclosure policy in a two-player contest

[5]Lu et al. (2018) define anonymous disclosure rules as policies that do not depend on the contestants' identities.

with incomplete information and binary effort. Melo-Ponce (2020) does not restrict the set of possible disclosure rules and, like us, finds that the optimal policy involves private and partial information revelation. Unlike us, however, he assumes that players have private information about their type, and finds that the optimal policy often requires asymmetric (and sometimes uncorrelated) signals. Such signals make a weak player more informed and incentivize him to exert higher effort. In our model, contestants are a priori identical, with no private information about their prize valuations. It is the designer who decides whether to endow them with private information, because she has a full control over the signals they receive. As a result, we find that disclosure rules that induce asymmetric beliefs are never optimal. This is different from the finding in Melo-Ponce (2020), though the intuition is somewhat similar: the highest efforts are achieved when the contestants have equal incentives to compete. Furthermore, our assumptions imply that the designer has a "first-mover advantage", which in the optimum allows her to extract the full surplus.[6]

Kuang et al. (2019) look at a two-player all-pay auction with continuous action space and correlated types. They allow for a rich set of disclosure rules, even though the signal distribution in their case cannot be state-dependent.[7] Most importantly and differently from our paper, the main part of the analysis in Kuang et al. (2019) is devoted to studying the case where both players know their true value of winning.[8] As we explained earlier, this limits the designer's control over the information the contestants have and leads to different results. In an extension, Kuang et al. (2019) allow for the situation where contestants do not know their valuations. But first, in this setting they assume that a signal from the designer about the valuations must always be perfectly precise, which, as we show, is not without loss of generality. And second, differently from us, they focus on the comparison of just three cases, where both, one or none of the players have private information on their winning values. Comparing these three cases, they find that from the designer's perspective, the setting where only one player knows his own type is dominated by the setting where no players know their own type.

[6] Also, in contrast to Melo-Ponce (2020), we cannot take advantage of the Bergemann and Morris (2016) approach because efforts in our model are not binary but continuous.

[7] Kuang et al. (2019) also employ a different timing where contestants first obtain a public and then private signal.

[8] Under this assumption, the authors formalize the ridge phenomenon of positive correlation and characterize the optimal disclosure policy. According to Kuang et al. (2019), the best disclosure rule never generates more than two posteriors, and one of them must be located at the ridge.

3 A simple example

To demonstrate the main idea and results, let us start with a simple example. Consider a research contest. Research contests or patent races are often used to spur innovation. Participants of such innovation races, who are commonly firms, compete to be the first to develop a new product and obtain a patent and monopoly rights to sell it. The value of being the winner is not always certain though. For example, the monopoly rents depend on future demand and the costs of producing a new good. These characteristics may vary vastly depending on market conditions, such as boom or recession, and exact features that the new product will gain in the outcome of conducted research. Thus, the value of winning for each contestant is determined by an unknown payoff-relevant state that reflects characteristics of both, the market and the contestants. In this example, it is convenient to think of a state as being multi-dimensional, where each dimension is firm-specific, that is, represents the value of winning for each competitor.

Neither the contestants nor the contest designer know the state perfectly, though they may hold a certain prior belief about it. However, the designer has a possibility to inquire more information about the state. For instance, she can solicit a report on the prospective market demand for each of the contestants' developed products or invite a committee of experts to evaluate submitted research proposals and their chances for market success. This information may then be revealed to the contestants, either privately or publicly, in an attempt to induce more research effort on their side.

In collecting the state-relevant information, the designer can select the quality of the inquiry, such as its precision.[9] However, a very precise report is helpful only if the realization of the state makes the contestants compete more aggressively (we call this a "good" state) and it is detrimental otherwise. Then, the designer could adjust the report to make it more precise in "good" than in "bad" states. For example, she can solicit a report that focuses on specific items that are likely to produce a favorable from her perspective outcome, which is precise only in good state. Finally, the designer can choose how to communicate the findings of the report to the contestants – privately, or publicly, or not at all.[10] If the

[9]For example, to increase the report's precision, the designer can choose many items to be evaluated or/and invite a very thorough consultant.

[10]As we make it explicit later, the designer cares only about the contestants' research efforts in the competition and does not derive any intrinsic value from the report itself. Therefore, she uses the report

designer committed to provide the information, then she must disclose the findings of the report truthfully. The question we are interested in is whether the designer who aims at maximizing the total research effort, can benefit from choosing the precision of the report and the way of communicating its findings to the contestants.

Let us now formalize this example. For simplicity, suppose that only two contestants, indexed by $i = 1, 2$, participate in the patent race. The value of winning for each contestant can be either high ($H = 4$) or low ($L = 1$). These are determined by realizations of i.i.d. random variables v_1 and v_2. Hence, there are four possible states of the world: (4, 4), (1, 1), (4, 1), and (1, 4). The designer and the contestants share a common prior belief about the state of the world, with $P(v_i = 4) = \frac{1}{2}$ for any $i = 1, 2$.

Since the contestants always extract a strictly positive value from winning the race, they have incentives to exert effort. Assume that the quality of the product they develop is proportional to the invested effort, and the highest quality producer wins. Then contestant i wins if and only if his effort is (weakly) higher than that of the rival, i.e., $e_i \geq e_{-i}$. The effort, however, is costly, and the contestants choose it by weighting their expected winning benefit against the effort cost, which we define as $\gamma(e_i) = e_i$.

To gain some intuition, consider a few baseline cases regarding the type of disclosure and report precision. Suppose the designer commits to share information publicly and solicits a report that is perfectly precise in every state.[11] Then, both contestants learn their true values of winning (v_1, v_2), and it turns the contest into a standard all-pay auction with complete information. The equilibrium of this game features mixed strategies of the following form (see Baye et al., 1996):

- If the contestants have the same value of winning $v \in \{1, 4\}$, then both of them randomize uniformly on the $[0,\ v]$ interval;
- If the values of winning are different, then the contestant with value $H = 4$ randomizes uniformly on the [0, 1] interval with probability 1, while his opponent with value $L = 1$ randomizes uniformly on the same interval with probability $\frac{L}{H} = \frac{1}{4}$ and stays inactive otherwise.

exclusively as a means of motivating the contestants. In particular, since we also assume that inquiring information is costless for the designer, not revealing information to contestants is equivalent to not soliciting a report.

[11] Perfectly precise report is not unrealistic and can be assumed without loss of generality if we interpret the state of the world as the most informative signal that can be produced.

With these equilibrium strategies, it is easy to rank the aggregate efforts over different states:

$$J_{HH} = 4 > J_{LL} = 1 > J_{HL} = J_{LH} = \frac{1}{2} \cdot \left(1 + \frac{1}{4}\right) = \frac{5}{8}$$

and this ranking is generic. Basically, the competition for a prize (a patent in this case) is strongest and prompts the highest effort when the contestants know that they have equal incentives to win, and it is particularly strong when these incentives are high. Since the designer chooses her disclosure policy (i.e., the report precision and the way to communicate its findings) *before* she learns the state of the world, she has to weigh each of these outcomes with a corresponding probability. Thus, from the designer's perspective, public disclosure with perfectly precise signals in each state results in the following *ex ante expected aggregate effort*:

$$\frac{1}{4} \cdot 4 + 2 \cdot \frac{1}{4} \cdot \frac{5}{8} + \frac{1}{4} \cdot 1 = \frac{25}{16}$$

This, in fact, is not very high because asymmetric states obtain a substantial weight. Intuitively, the designer could do better and increase the ex ante expected aggregate effort by "masking out" the possible asymmetry from the contestants. The simplest way to do this with public disclosure is to not reveal any information to the contestants in asymmetric states and send precise signals when the contestants' values of winning are identical. If the players receive no information, they symmetrically update their priors according to Bayes rule:

$$P_i \left(4 \middle| \varnothing\right) = \frac{P_{HL} I_{\{i=1\}} + P_{LH} I_{\{i=1\}}}{P_{HL} + P_{LH}} = \frac{1}{2}$$

and the expected value of winning turns out to be the same for both contestants:

$$E\left(v_i \middle| \varnothing\right) = \frac{1}{2} \cdot 4 + \frac{1}{2} \cdot 1 = \frac{5}{2}$$

As anticipated, the resulting ex ante expected aggregate effort is larger than the one under public disclosure with perfect precision in all states:

$$\frac{1}{4} \cdot 4 + 2 \cdot \frac{1}{4} \cdot \frac{5}{2} + \frac{1}{4} \cdot 1 = \frac{5}{2} = 2.5 > \frac{25}{16}$$

Thus, the designer indeed benefits from concealing information in asymmetric states. Note, however, that under such type of public disclosure, when contestants do not receive any signal, they know that their values are different. Would the aggregate effort increase even further if the designer decides to hide more information by sending private instead of public signals in all states? Let the designer commit to the following disclosure policy. All signals are to be transmitted to the contestants privately, and the precision of signals is chosen to be perfect if and only if the state is symmetric, that is, (4, 4) or (1, 1). In asymmetric states (4, 1) and (1, 4), the (signal generating) report

- reveals the true state with probability $\frac{2}{5}$,
- captures the value of exactly one contestant correctly (for example, sends (4, 4) or (1, 1) instead of the true state (4, 1)) with probability $\frac{1}{10}$, and
- reveals the "opposite" state (for example, (1, 4) instead of (4, 1)) with the remaining probability $\frac{2}{5}$.

With this disclosure rule, the report is noisy and features positive correlation between contestants' private messages. Each contestant learns only his own prize valuation – true in symmetric states and possibly false in asymmetric – but does not know the valuation of the other. Upon receiving a private signal m_i, each contestant updates his beliefs about own valuation and the opponent's signal as follows:[12]

$$P\left(v_i = 4 \middle| m_i = 4\right) = \frac{3}{4},\ P\left(v_i = 4 \middle| m_i = 1\right) = \frac{1}{4}$$

$$P\left(m_{-i} = 4 \middle| m_i = 4\right) = \frac{3}{5},\ P\left(m_{-i} = 4 \middle| m_i = 1\right) = \frac{2}{5}$$

Here, the first two probabilities define the belief of contestant i about his true valuation when the signal is m_i. The other two expressions characterize a probability that the opponent receives a "good" signal (namely, $m_{-i} = 4$), conditional on the information of contestant i. The resulting contest game is essentially a symmetric all-pay auction with incomplete information where a contestant's type is associated with the message received. The expected values of winning for each type are:

$$E\left(v_i \middle| m_i = 4\right) = 4 \cdot P\left(v_i = 4 \middle| m_i = 4\right) + 1 \cdot \left(1 - P\left(v_i = 4 \middle| m_i = 4\right)\right) = \frac{13}{4}$$

$$E\left(v_i \middle| m_i = 1\right) = 4 \cdot P\left(v_i = 4 \middle| m_i = 1\right) + 1 \cdot \left(1 - P\left(v_i = 4 \middle| m_i = 1\right)\right) = \frac{7}{4}$$

[12] The updating is symmetric, so we do not label the beliefs with index i.

In equilibrium:[13]

- a contestant who receives $m_i = 1$ randomizes uniformly on $[0, \underline{e}]$ with probability 1, where $\underline{e} = P\left(m_{-i} = 1 \middle| m_i = 1\right) \cdot E\left(v_i \middle| m_i = 1\right)$, and
- a contestant who receives $m_i = 4$ randomizes uniformly on $[\underline{e}, \bar{e}]$ with probability 1, where $\bar{e} = \underline{e} + P\left(m_{-i} = 4 \middle| m_i = 4\right) \cdot E\left(v_i \middle| m_i = 4\right)$.

It is easy to show that the ex ante expected aggregate effort in this case is equal to:

$$\frac{21}{20} \cdot \left(1 + \frac{1}{2}\right) + \frac{39}{20} \cdot \frac{1}{2} = \frac{102}{40} = 2.55$$

which is higher than under public disclosure. The intuition for this result is that such policy is best at making contestants believe that their values of winning are likely to be the same, which, as we saw, leads to the most intense competition. In the next sections, we generalize this example and provide a full characterization of the optimal disclosure policy.

4 The model

Two players striving to win a single prize engage in a contest game. The values of winning the prize can be different for the players and are determined by realizations of two i.i.d. discrete random variables v_i, $i = 1, 2$, where:

$$v_i = \begin{cases} H \text{ (high)}, & \text{with probability } \alpha \in (0, 1) \\ L \equiv 1 < H \text{ (low)}, & \text{with probability } 1 - \alpha \end{cases}$$

This results in four possible *states of the world*: (H, H), (L, L), (H, L), and (L, H).[14] Let us denote the set of these states by S. An important feature of our model is that neither the contest designer nor the contestants have any ex ante information about the realized state of the world and share a common symmetric prior, denoted by ξ.[15]

In the contest, players simultaneously choose non-negative effort e_i and pay the effort cost $\gamma_i(e_i) = e_i$ for any $i = 1, 2$. The player exerting higher effort wins and receives the prize, and

[13] See Liu and Chen (2016) and Section 5 for formal discussion of equilibria under private disclosure.

[14] Note that $\alpha = 0$ and $\alpha = 1$ are assumed away as otherwise, the question of information disclosure is irrelevant: the state is *known* to be (H, H) or (L, L), respectively.

[15] As should become clear from the model description below, the assumption about uninformed designer who draws signals about contestants' prize valuations with a predetermined precision and simply passes them on to the contestants, is equivalent to assuming that the designer holds the best possible knowledge about the state but prior to its realization commits to the precision of signals that she will send to the contestants.

ties are broken randomly. Thus, in the absence of any signal about the state of the world, player i's payoff is equal to

$$(\alpha H + (1-\alpha) L) P(e_i > e_{-i}) - e_i$$

where $(\alpha H + (1-\alpha) L)$ is the expected prize valuation of player i given the prior and $P(e_i > e_{-i})$ denotes the probability that player i wins. By making his effort choice e_i, player i maximizes this utility or, – if some information about valuations arrives, – a utility analogous to this, but calculated using an updated prior.

The designer aims at achieving the highest aggregate effort in the contest.As the contestants, she does not know the actual state of the world. But in contrast to them, she has access to a certain information technology that allows her to (costlessly) draw signals, or messages, $m_1, m_2 \in \{H, L\}$ about contestants' prize valuations v_1 and v_2. She can choose (1) the precision of this information technology, conditional on each state of the world, and (2) whether to pass the signals on to both contestants privately, or publicly, or keep them uninformed, with the goal of inducing maximal aggregate effort.

To be more precise, let us define an information technology $\mathcal{I} = (M, \pi, \tau)$ as a triple that consists of all possible signal profiles $M = \{(m_1, m_2) \text{ s.t. } m_i \in \{H, L\}, i = 1, 2\}$, conditional signal distributions $\pi : S \to \Delta(M)$, one for each possible state, and the type of information disclosure to contestants $\tau \in \{public, private, none\}$.[16] We assume that any type of disclosure in τ is applied to both players *symmetrically.* That is, either both players are informed about the signal publicly (each of them learns the same pair of messages (m_1, m_2)), or both are approached privately (player i observes only message m_i), or both remain uninformed and have to rely only on their prior. Furthermore, suppose that irrespective of whether the disclosure is public or private, the designer always conveys the information *truthfully.*

Since the contestants are symmetric, we restrict attention to signal distributions where conditional on every state of the world $\mathbf{s} \in S$, the probability that exactly one of the signals m_1, m_2 is different from the value in $\mathbf{s}$ is the same for both players. Such distributions can be fully characterized by two parameters per state $q_j, r_j \in [0, 1]$, $j \in 1:4$, as shown on Figure 1. Let us denote the resulting conditional signal distribution by $\pi = \pi\left(\{q_j, r_j\}_{j \in 1:4}\right)$.[17] The

[16] This concept is similar to the concept of "information structure" in Taneva (2019), where it is applied to a discrete-action game in which signals are action recommendations, and they are conveyed to players privately.

[17] Similar distributions but over action profiles and two states of the world emerge in Taneva (2019).

$\mathbf{s_1} = (H,\ H)$

	$m_2 = H$	$m_2 = L$
$m_1 = H$	r_1	$q_1 - r_1$
$m_1 = L$	$q_1 - r_1$	$1 - 2q_1 + r_1$

$\mathbf{s_2} = (H,\ L)$

	$m_2 = H$	$m_2 = L$
$m_1 = H$	$q_2 - r_2$	r_2
$m_1 = L$	$1 - 2q_2 + r_2$	$q_2 - r_2$

$\mathbf{s_3} = (L,\ H)$

	$m_2 = H$	$m_2 = L$
$m_1 = H$	$q_3 - r_3$	$1 - 2q_3 + r_3$
$m_1 = L$	r_3	$q_3 - r_3$

$\mathbf{s_4} = (L,\ L)$

	$m_2 = H$	$m_2 = L$
$m_1 = H$	$1 - 2q_4 + r_4$	$q_4 - r_4$
$m_1 = L$	$q_4 - r_4$	r_4

Figure 1: Conditional signal distribution π

parameter r_j indicates the probability with which signals about both players' prize valuations match the true state $\mathbf{s_j}$:

$$P(m_1 = H, m_2 = H|\mathbf{s_1} = (H,H)) = r_1,\ P(m_1 = H, m_2 = L|\mathbf{s_2} = (H,L)) = r_2,$$

$$P(m_1 = L, m_2 = H|\mathbf{s_3} = (L,H)) = r_3, P(m_1 = L, m_2 = L|\mathbf{s_4} = (L,L)) = r_4$$

In other words, r_j measures the probability that conditional on the state, the information technology reveals the true prize valuations of both contestants. The parameter q_j represents the probability that a signal regarding each contestant matches his actual valuation in state $\mathbf{s_j}$, irrespective of whether the same holds for a signal regarding the opponent. This probability is the same for both contestants. For example, consider state $\mathbf{s_2} = (H, L)$. Then, for contestant 1 this probability is given by

$$P(m_1 = H, m_2 = H|\ \mathbf{s_2} = (H,L)) + P(m_1 = H, m_2 = L|\ \mathbf{s_2} = (H,L)) = q_2$$

Similarly, for contestant 2, it is

$$P(m_1 = H, m_2 = L|\ \mathbf{s_2} = (H,L)) + P(m_1 = L, m_2 = L|\ \mathbf{s_2} = (H,L)) = q_2$$

Thus, parameters q_j and r_j measure the signal's precision, conditional on state $\mathbf{s_j}$: r_j reflects the precision of a signal about both contestants simultaneously, while q_j measures the precision of a signal about each contestant individually. Clearly, $q_j \geq r_j$ for any state $\mathbf{s_j}$.

The choice of q_j and r_j in each state also allows for arbitrary correlation between signals m_1 and m_2. In general, this correlation can be defined as

$$\rho = P(m_{-i} = H|m_i = H) - P(m_{-i} = H|m_i = L) = P(m_{-i} = L|m_i = L) - P(m_{-i} = L|m_i = H)$$

and it will be the focus of our attention when we study the case of private information disclosure. By calibrating the correlation between private messages that the designer sends to players, she can further her agenda of maximizing their aggregate effort.

Note that depending on the parameter values, the distribution over signal profiles π may, in fact, be independent of the state. This is the case when all four probability matrices on Figure 1 are identical, such as when all entries in each matrix are equal to 1/4, or when the same and equal weight is given only to symmetric signals (H, H), (L, L) in every state, and no weight is assigned to asymmetric signals. The former represents a situation where signals are completely uninformative, in that they do not affect the prior beliefs about players' prize valuations. Public revelation of such signals to contestants is equivalent to nondisclosure, or concealment. The latter represents an environment with the highest positive correlation between private signals. Moreover, the precision of these signals is zero if the realized state is asymmetric (namely, $r_2 = r_3 = 0$).

The timing of the events is as follows. Before the contest and any signal realization, the designer commits to and publicly announces an information technology I, which thus becomes common knowledge. Once the state has been randomly drawn from the prior distribution ξ, the signals are generated according to the announced distribution π and subsequently revealed to each player in line with the chosen type of disclosure τ. Upon observing the signal, each contestant i updates beliefs about the state and about the beliefs of his opponent. Then, both contestants simultaneously select a (possibly mixed) action – a probability distribution over effort levels – which maximizes their expected payoffs. The resulting choices conditional on players' signals define a Nash equilibrium or a Bayesian Nash equilibrium of the contest game. The exact solution concept depends on the type of information disclosure used. If the designer sends public messages or transmits no information ($\tau = \textit{public}$ or $\tau = \textit{none}$), the information set of both contestants is the same, and the appropriate equilibrium concept is Nash. In case of private disclosure ($\tau = \textit{private}$), the contest game features incomplete information, and each contestant can be of two types, H or L, as determined by his private message. Then, the appropriate concept is Bayesian Nash equilibrium.

In general, there could exist multiple equilibria of the contest game. The designer has to choose an information technology that induces contestants to play the equilibrium that maximizes her ex ante expected payoff, i.e., the ex ante expected aggregate effort of the contestants. This problem can be solved in two steps:

1. First, we characterize the set of equilibria that could emerge under any possible information technology, that is, under any type of information disclosure $\tau \in \{public, private, none\}$ and any conditional signal distribution $\pi\left(\{q_j, r_j\}_{j\in 1:4}\right)$.
2. Second, we maximize the objective function of the designer over the set of all possible equilibria. This delivers the optimal information technology I^* that consists of the optimal type of information disclosure τ^* and the optimal signal distribution $\pi(\{q_j^*, r_j^*\}_{j\in 1:4})$.

In what follows we adopt this approach. We first consider equilibria of the contest game under both public and private information disclosure. Then we solve the designer's problem in each case to derive an optimal signal distribution for $\tau = public$ and $\tau = private$. Nondisclosure, or information concealment ($\tau = none$), is addressed as a special case of public disclosure. Finally, we compare the optimal values of the designer's objective function across cases of public and private disclosure and characterize the global optimum, that is, the information technology I^* that the designer should optimally use in our model.

5 Equilibrium of the contest game

In this section, we characterize the equilibrium of the contest game under different disclosure regimes. If the designer adopts public disclosure, both contestants have the same information set, and the appropriate solution concept is a Nash equilibrium. With private disclosure, the contestants' information sets differ, and one must look for a Bayesian Nash equilibrium. We study these two cases separately.

5.1 Equilibrium under public information disclosure

Let us first consider the contest game following public information disclosure. To distinguish the notation in this case from the case of private disclosure (where additional restrictions will be imposed on distribution π), let us denote the precision parameters $\{q_j, r_j\}_{j\in 1:4}$ in this case by $\{q_j^{pub}, r_j^{pub}\}_{j\in 1:4}$.

Under public disclosure, the designer truthfully reveals the signal profile $\mathbf{m} = (m_1, m_2)$ to both players. Since the information received by the contestants is identical, they update beliefs about own and the opponent's prize valuations in the same way:[18]

$$P\left(v_i = H \middle| \, \mathbf{m} = (H,\, H)\,,\, \pi\right) =$$

$$= \frac{\alpha^2 r_1^{pub} + \alpha\left(1-\alpha\right)\left(\left(q_2^{pub} - r_2^{pub}\right) I_{\{i=1\}} + \left(q_3^{pub} - r_3^{pub}\right) I_{\{i=2\}}\right)}{\alpha^2 r_1^{pub} + \alpha\left(1-\alpha\right)\left(q_2^{pub} - r_2^{pub} + q_3^{pub} - r_3^{pub}\right) + \left(1-\alpha\right)^2\left(1 - 2q_4^{pub} + r_4^{pub}\right)}$$

$$P\left(v_i = H \middle| \, \mathbf{m} = (H,\, L)\,,\, \pi\right) =$$

$$= \frac{\alpha^2\left(q_1^{pub} - r_1^{pub}\right) + \alpha\left(1-\alpha\right)\left(r_2^{pub} I_{\{i=1\}} + \left(1 - 2q_3^{pub} + r_3^{pub}\right) I_{\{i=2\}}\right)}{\alpha^2\left(q_1^{pub} - r_1^{pub}\right) + \alpha\left(1-\alpha\right)\left(r_2^{pub} + 1 - 2q_3^{pub} + r_3^{pub}\right) + \left(1-\alpha\right)^2\left(q_4^{pub} - r_4^{pub}\right)}$$

$$P\left(v_i = H \middle| \, \mathbf{m} = (L,\, H)\,,\, \pi\right) =$$

$$= \frac{\alpha^2\left(q_1^{pub} - r_1^{pub}\right) + \alpha\left(1-\alpha\right)\left(\left(1 - 2q_2^{pub} + r_2^{pub}\right) I_{\{i=1\}} + r_3^{pub} I_{\{i=2\}}\right)}{\alpha^2\left(q_1^{pub} - r_1^{pub}\right) + \alpha\left(1-\alpha\right)\left(1 - 2q_2^{pub} + r_2^{pub} + r_3^{pub}\right) + \left(1-\alpha\right)^2\left(q_4^{pub} - r_4^{pub}\right)}$$

$$P\left(v_i = H \middle| \, \mathbf{m} = (L,\, L)\,,\, \pi\right) =$$

$$= \frac{\alpha^2\left(1 - 2q_1^{pub} + r_1^{pub}\right) + \alpha\left(1-\alpha\right)\left(\left(q_2^{pub} - r_2^{pub}\right) I_{\{i=1\}} + \left(q_3^{pub} - r_3^{pub}\right) I_{\{i=2\}}\right)}{\alpha^2\left(1 - 2q_1^{pub} + r_1^{pub}\right) + \alpha\left(1-\alpha\right)\left(q_2^{pub} - r_2^{pub} + q_3^{pub} - r_3^{pub}\right) + \left(1-\alpha\right)^2 r_4^{pub}}$$

where $I_{\{i=1\}}$ and $I_{\{i=2\}}$ are index functions equal to one if the condition in curly brackets holds, and zero otherwise. The expected prize valuation of player i, conditional on $\mathbf{m}$, is then given by:[19]

$$E\left(v_i \middle| \, \mathbf{m},\, \pi\right) \equiv v_i^{m,pub} = P\left(v_i = H \middle| \, \mathbf{m},\, \pi\right) H + P\left(v_i = L \middle| \, \mathbf{m},\, \pi\right)$$

Since the signal profile $\mathbf{m}$ is publicly observable, the contest mimics a complete information all-pay auction. This class of games was extensively studied in Baye et al. (1996). With two contestants, the Nash equilibrium is unique and features mixed strategies. Let's take any message $\mathbf{m} \in M$ and assume $v_i^{m,pub} \geq v_{-i}^{m,pub}$, that is, $P\left(v_i = H \middle| \, \mathbf{m},\, \pi\right) \geq P\left(v_{-i} = H \middle| \, \mathbf{m},\, \pi\right)$. Then, in the Nash equilibrium:[20]

- Contestant i randomizes uniformly on $\left[0,\, v_{-i}^{m,pub}\right]$ with probability 1,

[18] All probabilities are calculated by Bayes rule using parameters of the conditional distribution $\pi\left(\{q_j^{pub}, r_j^{pub}\}_{j\in 1:4}\right)$ (see Figure 1) and the prior distribution ξ defined in the previous section.

[19] Recall that $L \equiv 1$.

[20] See Baye et al. (1996) for technical details.

- Contestant $-i$ randomizes uniformly on $\left(0,\, v_{-i}^{m,pub}\right]$ with probability $\frac{v_{-i}^{m,pub}}{v_{i}^{m,pub}}$ and stays inactive with probability $\left(1-\frac{v_{-i}^{m,pub}}{v_{i}^{m,pub}}\right)$.

Note that given these strategies, the contestants' expected equilibrium payoffs are:

$$\begin{aligned} u_i &= \frac{v_{-i}^{m,pub}}{v_{i}^{m,pub}}\cdot\frac{1}{2}\cdot v_{i}^{m,pub}+\left(1-\frac{v_{-i}^{m,pub}}{v_{i}^{m,pub}}\right)\cdot v_{i}^{m,pub}-\frac{v_{-i}^{m,pub}}{2}=v_{i}^{m,pub}-v_{-i}^{m,pub}\geq 0 \\ u_{-i} &= \frac{v_{-i}^{m,pub}}{v_{i}^{m,pub}}\cdot\frac{1}{2}\cdot v_{-i}^{m,pub}-\frac{v_{-i}^{m,pub}}{v_{i}^{m,pub}}\cdot\frac{v_{-i}^{m,pub}}{2}=0 \end{aligned}$$

Furthermore, we can define J^{pub}, the ex ante expected aggregate effort of the contestants, where "ex ante" refers to the fact that the designer commits to the information technology *before* the realization of a signal:

$$J^{pub}\equiv\sum_{\mathbf{m}\in\tilde{M}_{\pi}}P\left(\mathbf{m}\right)\frac{v_{2}^{m,pub}}{2}\left(1+\frac{v_{2}^{m,pub}}{v_{1}^{m,pub}}\right)+\sum_{\mathbf{m}\notin\tilde{M}_{\pi}}P\left(\mathbf{m}\right)\frac{v_{1}^{m,pub}}{2}\left(1+\frac{v_{1}^{m,pub}}{v_{2}^{m,pub}}\right), \qquad (1)$$

where $\tilde{M}_{\pi}\subseteq M$ denotes the set of all such signal profiles $\mathbf{m}$ for which $v_{1}^{m,pub}\geq v_{2}^{m,pub}$.

5.2 Equilibrium under private information disclosure

Consider now the case of private information disclosure. Upon observing signal profile $\mathbf{m}=(m_1,m_2)$, the designer truthfully reveals m_1 to contestant 1 and m_2 to contestant 2. Thus, private signals create an environment with asymmetric information, where each contestant's message m_i is his privately observed type. To simplify things, let us assume in the analysis of private information disclosure that conditional on realization of either of the asymmetric states (H,L) or (L,H), the signal distribution π is characterized by the same precision parameters. This reduces the number of choice variables for the designer and, hence, the dimensionality of the optimization problem she has to solve. Figure 2 depicts the signal distribution for this case.

Furthermore, we employ a standard Bayesian rationality requirement that the distribution of contestants' beliefs is Bayes-plausible. By definition (see Kamenica and Gentzkow (2011)), this means that for every player i, the expected posterior probability equals the prior, that is, the probability of receiving signal $m_i=H$ is equal to α. We will say that the signal distribution π that induces such Bayes-plausible beliefs is *consistent*.

$\mathbf{s_1} = (H,\, H)$

	$m_2 = H$	$m_2 = L$
$m_1 = H$	r_1	$q_1 - r_1$
$m_1 = L$	$q_1 - r_1$	$1 - 2q_1 + r_1$

$\mathbf{s_2} = (H,\, L)$

	$m_2 = H$	$m_2 = L$
$m_1 = H$	$q_2 - r_2$	r_2
$m_1 = L$	$1 - 2q_2 + r_2$	$q_2 - r_2$

$\mathbf{s_3} = (L,\, H)$

	$m_2 = H$	$m_2 = L$
$m_1 = H$	$q_2 - r_2$	$1 - 2q_2 + r_2$
$m_1 = L$	r_2	$q_2 - r_2$

$\mathbf{s_4} = (L,\, L)$

	$m_2 = H$	$m_2 = L$
$m_1 = H$	$1 - 2q_3 + r_3$	$q_3 - r_3$
$m_1 = L$	$q_3 - r_3$	r_3

Figure 2: Conditional signal distribution π in case of private information disclosure

Definition. A disclosure rule is consistent if and only if $P(m_i = H) = \alpha$ for $i = 1, 2$, that is, $\alpha^2 q_1 + \alpha(1-\alpha) + (1-\alpha)^2 (1 - q_3) = \alpha$.

Note that the consistency condition holds for any $\alpha \in (0, 1)$, when the disclosure of symmetric states is perfectly precise: $q_1 = q_3 = 1$, so that also $r_1 = r_3 = 1$. However, the condition also holds for other q_1 and q_3, provided that α is defined appropriately. Interestingly, our results in the next section imply that despite these restrictions on signal distributions, the optimal information disclosure with private signals ($\tau = private$) delivers higher expected aggregate effort than the best public disclosure rules ($\tau = public$ and $\tau = none$).

Having received signal m_i from the designer, each player i updates his beliefs about own valuation and the opponent's type m_{-i} as follows:[21]

$$P(v_i = H |\, m_i = H,\, \pi) \;=\; \frac{\alpha^2 q_1 + \alpha(1-\alpha) q_2}{\alpha} \tag{2}$$

$$P(v_i = H |\, m_i = L,\, \pi) \;=\; \frac{\alpha^2 (1 - q_1) + \alpha(1-\alpha)(1 - q_2)}{1 - \alpha} \tag{3}$$

and

$$P(m_{-i} = H |\, m_i = H,\, \pi) \;\equiv\; P_{H|H} = \frac{\alpha^2 r_1 + 2\alpha(1-\alpha)(q_2 - r_2) + (1-\alpha)^2 (1 - 2q_3 + r_3)}{\alpha}$$

$$P(m_{-i} = H |\, m_i = L,\, \pi) \;\equiv\; P_{H|L} = \frac{\alpha^2 (q_1 - r_1) + \alpha(1-\alpha)(1 - 2q_2 + 2r_2) + (1-\alpha)^2 (q_3 - r_3)}{1 - \alpha}$$

To complete the notation, let us define $P_{L|H} = 1 - P_{H|H}$ and $P_{L|L} = 1 - P_{H|L}$. Beliefs in (2)–(3) determine the expected prize valuation of player i, conditional on message m_i:

[21] These probabilities are computed by Bayes rule using the conditional distribution π in Figure 2 and the prior distribution ξ.

$$E\left(v_i|\, m_i,\, \pi\right) \equiv v_i^m = P\left(v_i = H|\, m_i,\, \pi\right) H + P\left(v_i = L|\, m_i,\, \pi\right)$$

From this equation it is clear that the type of player i determined by message m_i can be equivalently defined by his expected valuation v_i^m:

$$m_i = H \text{ if and only if } v_i^m = v_i^H$$

Moreover, since $v_1^H = v_2^H$ and $v_1^L = v_2^L$ hold, we can drop the player subscript i. Therefore, we will refer to v_i^H and v_i^L as *types* v^H and v^L of player i.

In the analysis, we focus on the case where $v^H \geq v^L$. This describes a situation where receiving signal H means good news and results in higher expected valuation. In other words, we require the designer to stick to the signal distributions that preserve the relationship between valuations H and L. Since $H > L$, the inequality $v^H \geq v^L$ is equivalent to $P\left(v_i = H|\, m_i = H,\, \pi\right) \geq P\left(v_i = H|\, m_i = L,\, \pi\right)$, that is, $q_1 \geq 1 - \frac{1-\alpha}{\alpha} q_2$.

To characterize the contestants' equilibrium behavior, we employ the results of Liu and Chen (2016). They provide a closed-form solution for both monotonic and non-monotonic symmetric Bayesian Nash equilibria of an all-pay auction with correlated types. In our setting, where types are players' expected valuations, or messages that they observe privately, the relevant correlation is between the contestants' private signals. It is inherent in the signal distribution π and equal to

$$\rho = P_{H|H} - P_{H|L} = P_{L|L} - P_{L|H}$$

where the second equality follows from $P_{L|H} = 1 - P_{H|H}$ and $P_{L|L} = 1 - P_{H|L}$. Similarly to Liu and Chen (2016), we define two types of equilibrium. If messages m_1 and m_2 are independent or mildly correlated (namely, $\rho = 0$ or slightly positive/negative), the equilibrium is monotonic. With sufficiently correlated messages ($\rho \gg 0$ or $\rho \ll 0$), the equilibrium becomes non-monotonic.[22] Each of the cases – with mild, sufficiently positive and sufficiently negative correlation – corresponds to respective intervals for $\frac{P_{L|L}}{P_{L|H}}$ and $\frac{P_{H|L}}{P_{H|H}}$. Indeed, $\frac{P_{L|L}}{P_{L|H}} = \frac{P_{H|L}}{P_{H|H}} = 1$ if and only if the contestants' types are independent, and any deviation from this corresponds to the presence of non-zero correlation. Using Proposition 1 of Liu and Chen (2016), we now describe the value of the ex ante expected aggregate effort that is generated in each equilibrium. The derivations are provided in the Appendix.

[22] Each equilibrium features mixed strategies. It is defined as monotonic if different types of a contestant randomize on non-overlapping intervals. Otherwise, the equilibrium is classified as non-monotonic.

Proposition 1. *If correlation between the two messages m_1 and m_2 is zero or mild, i.e., $\frac{v^H}{v^L} \geq \frac{P_{L|L}}{P_{L|H}}$ and $\frac{v^H}{v^L} \geq \frac{P_{H|L}}{P_{H|H}}$, the equilibrium is monotonic. The ex ante expected aggregate effort amounts to:*

$$J^M = (1+\alpha)\, P_{L|L} v^L + \alpha P_{H|H} v^H$$

If correlation between m_1 and m_2 is sufficiently positive, i.e., $\frac{v^H}{v^L} < \frac{P_{L|L}}{P_{L|H}}$, or sufficiently negative, i.e., $\frac{v^H}{v^L} < \frac{P_{H|L}}{P_{H|H}}$, the equilibrium is non-monotonic. The ex ante expected aggregate effort in case of positive correlation is equal to:

$$J^{NM,+} = \frac{v^H \left(v^L \left(P_{L|L} - P_{L|H}\right) + \alpha \left(v^H - v^L\right)\right)}{P_{H|H} v^H - P_{H|L} v^L}$$

and in case of negative correlation it is equal to:

$$J^{NM,-} = \frac{v^L \left(v^H - v^L\right) \left(P_{L|L} - (1-\alpha)\right)}{P_{L|H} v^H - P_{L|L} v^L} + v^L$$

In all equilibria, a contestant of type L gets zero expected payoff:

$$u_L^M = u_L^{NM,+} = u_L^{NM,-} = 0$$

while a contestant of type H receives an expected payoff given by:

$$u_H^M = P_{L|H} v^H - P_{L|L} v^L,$$

$$u_H^{NM,+} = 0, \;\; u_H^{NM,-} = v^H - v^L$$

Note that Proposition 1 does not immediately imply the ranking of the three equilibria in terms of the expected aggregate effort they generate. This depends on the designer's choice of the conditional signal distribution π, that is, on parameters $\{q_j,\, r_j\}_{j=1}^3$ which parametrize it (see Figure 2). In our search for the optimal private disclosure rule we will proceed in two steps. First, we will study an optimal choice of the signal distribution π for each of the equilibria separately. Second, we will compare the expected aggregate effort associated with each choice and define the best signal distribution under private information disclosure.

6 The optimal signal distribution

In this section, we use a mixture of analytical and numerical methods to solve for the optimal signal distribution. First, we state a general result that does not depend on the type of information disclosure τ and illustrates a typical first-mover advantage. Since the contestants are ex ante identical and have no private information, the designer, who commits to the signal distribution before the competition starts, can extract full surplus.

Lemma 1. *For any type of information disclosure, an optimal information technology leaves no surplus to either of the contestants, i.e., $u_i = 0$ for any $i = 1, 2$.*

We prove this result by contradiction. Suppose that the optimal disclosure rule is such that $u_i > 0$ for some contestant i and some message profile $\mathbf{m} \in M$. Since the expected equilibrium payoff of the contestant with lower prize valuation is always zero, contestant i's valuation is higher, i.e., $v_i^m > v_{-i}^m$. Then, in the absence of ex ante heterogeneity and private information on the contestants' side, the designer can slightly perturb the signal distribution in a way that makes v_{-i}^m larger (but still below v_i^m), so that the expected effort of $-i$, conditional on message $\mathbf{m}$, increases.[23] This means that the expected payoff of contestant i declines: indeed, even if i reacts to the change by increasing his effort just enough to keep the probability of winning the same, his effort cost increases. Since there is no other party that could capture the contestants' surplus, and externalities are absent, the reduction in $u_i > 0$ benefits the designer.[24] This contradicts the assumption that the designer's information technology is optimal.

Equipped with Lemma 1, we can now characterize an optimal signal distribution under public disclosure. From Section 5.1 it follows that the expected equilibrium payoffs of the contestants in this case are equal to:

$$u_i = \left(v_i^{m,pub} - v_{-i}^{m,pub}\right) \cdot I_{\left\{v_i^{m,pub} \geq v_{-i}^{m,pub}\right\}} \quad \forall\, \mathbf{m} \in M,\, i = 1, 2$$

where $I_{\left\{v_i^{m,pub} \geq v_{-i}^{m,pub}\right\}}$ is an index function equal to one if and only if the condition $v_i^{m,pub} \geq v_{-i}^{m,pub}$ holds and zero otherwise. Then, according to Lemma 1, an optimal signal distribution must induce $v_1^{m,pub} = v_2^{m,pub}$ for any message $\mathbf{m} \in M$. Now, the equality $v_1^{m,pub} = v_2^{m,pub}$ (or alternatively, $P\left(v_1 = H|\, \mathbf{m}\right) = P\left(v_2 = H|\, \mathbf{m}\right)$) is true for any signal $\mathbf{m} \in M$ if and only if

[23] This perturbation can keep the surplus of player $-i$ unaffected by balancing the two components of the contestant's payoff.

[24] For any information technology, contestants' expected effort is always positive because no matter what the state is, they attach a value of at least $L \equiv 1$ to the prize, i.e. winning is always attractive.

the following three conditions hold:[25]

$$q_2^{pub} - r_2^{pub} = q_3^{pub} - r_3^{pub} \tag{4}$$

$$r_2^{pub} = 1 - 2q_3^{pub} + r_3^{pub} \tag{5}$$

$$r_3^{pub} = 1 - 2q_2^{pub} + r_2^{pub} \tag{6}$$

Thus, these conditions are necessary for an optimal signal distribution π under public disclosure. Moreover, as we show in the proof of Proposition 2, they are also sufficient for an optimum because the designer's objective function achieves its upper bound on the set of feasible parameters when these conditions hold.

Proposition 2. *Under public disclosure (τ = public), an optimal signal distribution $\pi\left(\{q_j^{pub}, r_j^{pub}\}_{j\in 1:4}\right)$ is characterized by conditions* (4)–(6)*, that is, it generates symmetric beliefs, where $P\left(v_1 = H|\,\mathbf{m}\right) = P\left(v_2 = H|\,\mathbf{m}\right)\ \forall\,\mathbf{m} \in M$. The ex ante expected aggregate effort associated with the optimal signal distribution is equal to $J^{pub} = \alpha H + (1-\alpha)$.*

The interpretation of this result is the following. By definition of distribution π in Figure 1, conditions (4)–(6) make the probability matrices corresponding to states $\mathbf{s_2} = (H, L)$ and $\mathbf{s_3} = (L, H)$ identical, without imposing any further restrictions on the probability matrices for states $\mathbf{s_1} = (H, H)$ and $\mathbf{s_4} = (L, L)$. This means that for any signal realization (m_1, m_2), the asymmetric states (H, L) and (L, H) receive equal weight in contestants' beliefs. The result confirms our intuition from the opening example: if contestants believe that their prize valuations are different, they exert the lowest possible effort. To prevent this, the designer should obscure the information about such states. In particular, an information technology with precise revelation of *every* state is never optimal, because at least in asymmetric states, the designer would prefer the signal to be imprecise.

Another immediate observation is that the optimal signal distribution under public disclosure is not unique. In fact, any distribution that results in symmetrically updated beliefs, where for any signal $\mathbf{m}$ each contestant is equally likely to have a high prize valuation, is optimal. In the Appendix, we provide two examples of optimal public disclosure rules. The first one is equivalent to complete nondisclosure, or concealment. It implies that under public disclosure regimes, not revealing any information to contestants is

[25]This follows immediately from the definition of probabilities $P\left(v_i = H|\,\mathbf{m},\,\pi\right)$ in the beginning of Section 5.1 and the assumption that $\alpha \in (0, 1)$.

one optimal possibility. The second example emphasizes that precise revelation can benefit the designer only in symmetric states (H, H) and (L, L), while any disclosure in case of asymmetric prize valuations must be uninformative.

However, it turns out that even the optimal public disclosure regime is never globally optimal, as there exists a private disclosure rule that delivers a higher payoff to the designer.

Proposition 3. *Public information disclosure is never optimal.*

To prove this proposition, we construct a signal distribution with private signals that generates a higher ex ante expected aggregate effort than the best signal distribution under public disclosure. This result is sufficient to show that the optimal information technology I^* must feature private signals. Now, to uncover the properties of the optimal signal distribution π^*, we take a closer look at the private signals case.

6.1 Private information disclosure

We start from studying an optimal choice of a signal distribution π for each of the equilibria in the contest game under private disclosure separately (see Proposition 1). First, we derive an optimal signal distribution that supports the monotonic equilibrium. Then, we study the properties of optimal distributions that induce non-monotonic equilibria with positive and negative correlation. After that we compare the expected aggregate effort associated with each choice and define the best signal distribution under private information disclosure. In view of Proposition 3, this will determine the global optimum in our model. We will see that at least as long as H is not much larger than $L \equiv 1$, the globally optimal signal distribution induces the monotonic equilibrium with slightly positively correlated private signals that are perfectly precise if and only if the state is symmetric.

6.1.1 An optimal signal distribution supporting the monotonic equilibrium

An optimal signal distribution that supports the monotonic equilibrium is a solution to the following maximization problem:

$$\max_{\pi=\{q_j, r_j\}_{j=1}^3} \left\{ J^M \equiv (1+\alpha)\, P_{L|L} v^L + \alpha P_{H|H} v^H \right\} \tag{7}$$

$$\text{s. t. } \alpha^2 q_1 + \alpha(1-\alpha) + (1-\alpha)^2(1-q_3) = \alpha \tag{8}$$

$$q_1 \geq 1 - \frac{1-\alpha}{\alpha} q_2 \tag{9}$$

$$v^H P_{L|H} - v^L P_{L|L} \geq 0 \tag{10}$$

$$v^H P_{H|H} - v^L P_{H|L} \geq 0 \tag{11}$$

$$q_j \geq r_j \tag{12}$$

$$r_j \geq \max\{0,\, 2q_j - 1\} \tag{13}$$

$$q_j,\, r_j \in [0,\, 1],\, j = 1,\, ..,\, 3 \tag{14}$$

The set of constraints includes the consistency requirement in (8), the inequality stating that $v^H \geq v^L$ in (9),[26] two "monotonicity" conditions to sustain a monotonic equilibrium in (10)–(11), and the feasibility constraints ensuring that the signal distribution π is well defined in (12)–(14). Due to non-linearity and multi-dimensionality of this constrained optimization problem, finding all solutions analytically does not appear feasible. However, Lemma 2 formulates one general property that any optimal private disclosure policy inducing J^M must possess.

Lemma 2. *Under private disclosure, an optimal signal distribution that supports a monotonic equilibrium must satisfy $P_{L|H}v^H = P_{L|L}v^L$, i.e., condition* (10) *binds.*

The proof follows immediately if we recall from Lemma 1 that the expected payoff of both contestant types must be zero, and $u_H^M = P_{L|H}v^H - P_{L|L}v^L$, as stated by Proposition 1. Lemma 2 implies that private signals of the contestants must be (mildly) positively correlated. This benefits the designer as it makes the contestants give less weight to asymmetric states, and the expected aggregate effort increases.

We now analytically derive one (local) optimum for the case when the prior probability α of having a high valuation is above a certain threshold. Then, we consider a candidate for an optimum when α lies below that threshold. Later we will look for other possible solutions numerically and demonstrate that the analytically derived optima are, in fact, global for a large range of parameter values. Specifically, deviations to a different disclosure policy turn

[26] Recall that this is the assumption we made in section 5.2.

out to be profitable only when valuations H and L are sufficiently different and α is small enough.

Proposition 4 states that for $\alpha \geq \max\left\{0, \frac{H-3}{3(H-1)}\right\}$, an optimal signal distribution that supports the monotonic equilibrium is such that the symmetric states (H, H) and (L, L) are revealed precisely, whereas the asymmetric states are reported with a noise. Moreover, conforming to Lemma 2, the first monotonicity constraint in (10) binds, so that the contestants' messages are slightly but positively correlated.

Proposition 4. *For $\alpha \geq \max\left\{0, \frac{H-3}{3(H-1)}\right\}$, there exists a (locally) optimal signal distribution $\pi\left(\{q_j, r_j\}_{j\in 1:3}\right)$ that supports a monotonic equilibrium under private disclosure ($\tau =$ private). It is characterized by such precision parameters $\{q_j, r_j\}_{j\in 1:3}$ that generate a small but strictly positive correlation between private messages (condition (10) binds), perfectly precise signals in case of the symmetric states (H, H) and (L, L) and imprecise signals in case of the asymmetric states (H, L) and (L, H). Specifically:*

$$q_1 = r_1 = q_3 = r_3 = 1$$

$$q_2 = \min\{\hat{q}_2, \bar{q}_2\}$$

$$r_2 = q_2 - \frac{(1-\alpha)\left(v^H - v^L\right)}{2\left((1-\alpha)v^H + \alpha v^L\right)} \equiv \hat{r}_2(q_2) > 0$$

where

$$\bar{q}_2 = \begin{cases} \frac{H-3-5\alpha(H-1)-\sqrt{D}}{4(H-1)(1-2\alpha)}, & \alpha \neq \frac{1}{2} \\ \frac{2(H+1)}{3H+1}, & \alpha = \frac{1}{2} \end{cases}$$

$$\hat{q}_2 = \begin{cases} \frac{\alpha-\sqrt{\alpha(1-\alpha)}(1+\alpha(H-1))}{\alpha(2\alpha-1)(H-1)}, & \alpha \neq \frac{1}{2} \\ \frac{H+1}{2(H-1)}, & \alpha = \frac{1}{2} \end{cases}$$

$$D = -7(H-1)\alpha^2 + 6\left(H^2 - 4H + 3\right)\alpha + H^2 + 10H - 7 > 0 \,\forall\, \alpha \in [0, 1]$$

Intuitively, and in line with the opening example and our results on public disclosure, the designer has an incentive to keep the competition between contestants even. This is achieved by perfectly revealing the symmetric states and adding noise to the information when the state is asymmetric. Positive correlation between contestants' messages is also a step in this direction, particularly given that the prior is strong (α is sufficiently high), that is, high prize valuations are likely.[27]

[27] Recall that the highest competitive effort is exerted when contestants know that both of them have high

Now, to gain some insights into how an optimal disclosure rule may look like for a range of lower values of α, we compare the optimal signal distribution in Proposition 4 with its counterpart under which the asymmetric states (H, L) and (L, H) are *never* revealed precisely, that is, where $r_2 = 0$. We show that for such a distribution, there exists a range of lower values of α and a range of sufficiently different H and $L \equiv 1$, such that the resulting ex ante expected aggregate effort is higher than under the optimal policy of Proposition 4. To make the statement precise, let us denote by $J^M(q_1, r_1, q_2, r_2, q_3, r_3)$ the value of the ex ante expected aggregate effort (in the monotonic equilibrium) when the precision parameters are fixed at $\{q_i,\, r_i\}_{i=1}^3$.

Proposition 5. *Assume $q_1 = r_1 = q_3 = r_3 = 1$ and monotonicity condition (10) binds. Then, for $H \geq \left(9 + 4\sqrt{5}\right)$, there exist $\tilde{\alpha}_1 > 0$ and $\tilde{\alpha}_2 \in \left(\tilde{\alpha}_1,\, \frac{H-3}{3(H-1)}\right)$ such that for any $\alpha \in (\tilde{\alpha}_1,\, \tilde{\alpha}_2)$, a counterpart of the signal distribution in Proposition 4 where the asymmetric states $(H,\, L)$ and $(L,\, H)$ are never revealed precisely ($r_2 = 0$) induces a higher ex ante expected aggregate effort:*

$$\begin{cases} H \geq \left(9 + 4\sqrt{5}\right) \\ \tilde{\alpha}_2 \in \left(\tilde{\alpha}_1,\, \frac{H-3}{3(H-1)}\right) \end{cases} \Rightarrow$$

$$J^M\left(1,\, 1,\, \underline{q}_2,\, 0,\, 1,\, 1\right) > J^M\left(1,\, 1,\, \min\{\hat{q}_2,\, \bar{q}_2\},\, \hat{r}_2\left(\min\{\hat{q}_2,\, \bar{q}_2\}\right),\, 1,\, 1\right)$$

where

- $\tilde{\alpha}_{1,2} = \frac{H-9\mp\sqrt{H^2-18H+1}}{10(H-1)}$,
- $\underline{q}_2 = \frac{H-3-3\alpha(H-1)}{2(1-2\alpha)(H-1)}$, *and*
- $\hat{q}_2$, $\bar{q}_2$, *and* $\hat{r}_2(q_2)$ *are defined in Proposition 4.*

While this proposition does not claim optimality of the described policy with $r_2 = 0$, it hints that it may dominate other disclosure rules. In the next subsection and in section 6.1.3 we will study alternative local optima under private disclosure, that support both monotonic and non-monotonic equilibria of the contest game. We will see that the signal distributions of Propositions 4 and 5 do indeed generate the highest expected aggregate effort for a broad range of α and H.

valuation.

6.1.2 Optimal signal distributions supporting the non-monotonic equilibria

To understand the properties of the optimal signal distributions that induce non-monotonic equilibria, let us again refer to Lemma 1. First, consider the case of sufficiently negative correlation between contestants' messages, that is, $\frac{v^H}{v^L} < \frac{P_{H|L}}{P_{H|H}}$. By Proposition 1, the expected equilibrium payoff of a contestant of type H is $u_H^{NM,-} = v^H - v^L$. Combining this with Lemma 1, we obtain the following result.

Lemma 3. *Under private disclosure, an optimal signal distribution that supports a non-monotonic equilibrium with negative correlation must satisfy* $v^H = v^L$*, i.e., condition* (9) *has to bind.*

Lemma 3 implies that with negatively correlated messages, the meaning of private signals gets lost, and the case becomes equivalent to public disclosure. In particular, one can show that the ex ante expected aggregate effort of the contestants in this case is exactly the same as under the optimal signal distribution in case of public disclosure, i.e., $J^{NM,-} = \alpha H + (1-\alpha)$. Indeed, since the optimal signal distribution must satisfy $v^H = v^L$, the ex ante expected aggregate effort $J^{NM,-}$ reduces to (see Proposition 1):

$$J^{NM,-} = \frac{v^L\left(v^H - v^L\right)\left(P_{L|L} - (1-\alpha)\right)}{P_{L|H}v^L - P_{L|L}v^L} + v^L = v^L$$

By substituting $q_1 = 1 - \frac{1-\alpha}{\alpha}q_2$ (which is condition (9) when it binds) into $J^{NM,-} = v^L$, we obtain:

$$J^{NM,-} = \alpha H + (1-\alpha) = J^{pub}$$

The intuition behind this result is the following. If a signal distribution features sufficiently negative correlation, each contestant believes that most likely, the opponent is of a different type. As a result, both competitors assign more weight to asymmetric signals, and their efforts decline. To mitigate this effect, the best the designer can do is to design a signal distribution that induces $v^H = v^L$, independently of the exact signal realizations. This equivalence with the case of public disclosure, together with Proposition 3, lead to the following result:

Proposition 6. *Private disclosure with strongly negatively correlated signals is never optimal.*

As for the case with sufficiently positively correlated messages, where $\frac{v^H}{v^L} < \frac{P_{L|L}}{P_{L|H}}$, unfortunately, we cannot say much. By Proposition 1, for *any* signal distribution π that satisfies $\frac{v^H}{v^L} < \frac{P_{L|L}}{P_{L|H}}$, both contestants' types receive zero expected payoff. However, such signal distributions do not need to be equivalent in terms of the expected aggregate effort they generate. Also, the corresponding optimization program of the designer turns out to be analytically intractable. To overcome these issues, in the next subsection we perform numerical simulations to see whether a signal distribution that generates the non-monotonic equilibrium with positively correlated private signals can be optimal.

Intuitively, we expect that the ex ante expected aggregate effort of the contestants in this case is lower than in the monotonic equilibrium. To see this, let us first consider the strategies played by the contestants in the non-monotonic equilibrium with positive correlation:[28]

- Type L randomizes uniformly on $[0, \underline{e}]$ with probability 1, where $\underline{e} \in \left(0, v^H\right)$,
- Type H randomizes on the same interval with probability $p_B^H = \frac{P_{L|L}v^L - P_{L|H}v^H}{P_{H|H}v^H - P_{H|L}v^L}$ and exhausts the remaining probability in $\left[\underline{e}, v^H\right]$.

Using this, we can rewrite the ex ante expected aggregate effort $J^{NM,+}$ (see Proposition 1) as follows:

$$J^{NM,+} = \underline{e} + \alpha v^H (1 - p_B^H)$$

Since $J^{NM,+}$ strictly decreases in p_B^H, it is intuitive that the designer should reduce this probability and, hence, prompt type H to play in the top interval more often. This would bring the equilibrium play closer to the one in the monotonic equilibrium, where type H chooses his effort in a strictly higher interval than type L. Also, $p_B^H = 0$ holds in the monotonic equilibrium, under the optimal signal distribution (where condition (10) binds). This suggests that the designer may benefit from moving as close to the monotonic equilibrium as possible.[29]

Another indication of that is an observation that sending perfectly correlated private signals cannot do better than the optimal public disclosure rule, which, in turn, is dominated by the optimal private disclosure inducing the monotonic equilibrium. The second part of

[28] See Proposition 1 of Liu and Chen (2016).

[29] The fact that $J^{NM,+}$ strictly decreases in p_B^H is not sufficient to prove that the designer will necessarily improve $J^{NM,+}$ by choosing parameters of the signal distribution π so as to set $p_B^H = 0$. This is because the same parameters enter other components of $J^{NM,+}$ – $\underline{e}$ and v^H, – and the effect on them can be suboptimal.

that statement is established in the proof of Proposition 3. Here we show that the expected aggregate effort under perfectly correlated private signals is equal to the one under optimal public disclosure. Under $\tau = \mathit{private}$, the following signal distribution supports perfect correlation, $\rho = 1$:

$$r_1 = q_1 = r_3 = q_3 = 1$$

$$r_2 = 0,\ q_2 = \frac{1}{2}$$

This induces $P_{H|H} = P_{L|L} = 1$, $v^H = H$, and $v^L = 1$. Then one can easily show that

$$J^{NM,+} = \alpha H + (1-\alpha) \text{ and } p_B^H = \frac{1}{H}$$

Thus, $J^{NM,+}$ coincides with the expected aggregated effort under optimal public disclosure (see Proposition 2).

Two opposing forces lead to the tradeoff that the designer faces when she generates the non-monotonic equilibrium with perfectly positively correlated signals. First, the proposed signal distribution induces an equilibrium with the largest possible support of contestants' strategies, $e \in [0,\, H]$. Second, the probability p_B^H that type H plays in the lower interval $[0,\, \underline{e}]$ is sufficiently high. The former works in the direction of increasing the expected aggregate effort, while the latter works in the opposite direction. This tradeoff demonstrates the difficulty with achieving the highest possible aggregate effort despite perfectly aligned messages of the contestants.

Altogether, the results of our analysis so far imply that the optimal information technology has $\tau^* = \mathit{private}$ with non-negatively correlated private signals. Moreover, since a signal distribution that features *perfect* positive correlation cannot do better than the optimal public disclosure, it must be that the optimal signal distribution induces either the monotonic equilibrium with slightly positively correlated private signals or the non-monotonic equilibrium where the correlation between signals is positive but not too strong. To take a deeper look at these issues and find a globally optimal signal distribution in our model, we perform numerical simulations.

6.1.3 Global optimum via simulations

In this section we present the results of numerical simulations and check if there exist private disclosure rules that outperform the policies characterized in Propositions 4 and 5. The best of these rules will constitute the global optimum in our model.

Let us denote by M and NM^+ the best information disclosure policies that support, respectively, the monotonic equilibrium and the non-monotonic equilibrium with strong positive correlation between private signals. In simulations, we vary the degree of heterogeneity in prize valuations, $H/L \equiv H$, and the prior probability of a high prize valuation, α. For each (α, H) pair, we find the optimal policies M and NM^+, and then choose the best of them, that is, the policy that generates the highest expected aggregate effort.

Figure 3 shows when policy M is a global optimum. As one can see, for any (α, H) pair, the best disclosure rule supports the monotonic equilibrium. With this policy the designer sends mildly positively correlated private messages, signaling the contestants that they are not too different, and achieves the highest ex ante expected aggregate effort. Thus, we can formulate the following result.

Result 1. *The optimal information technology features private signal disclosure, $\tau^* = private$, and induces the monotonic equilibrium in the contest game.*

Next, we take a closer look at the optimal disclosure policies that support the monotonic equilibrium. Figure 4 demonstrates how a specific form of policy M depends on α and H. As one can see, in case of mild ($H = 2$ and $H = 12$) and moderate ($H = 22$) heterogeneity in prize valuations, the policy with $q_1 = r_1 = q_3 = r_3 = 1$ is always the best and, hence, constitutes the global optimum. For strong heterogeneity ($H = 32$), however, there exists a non-empty set of α's, $\alpha \in [0.09, 0.13]$, for which disclosure rules with $q_1 = r_1 = q_3 = r_3 = 1$ are no longer optimal. Table 1 reports the best information revelation policies when this is the case. Moreover, whenever $q_1 = r_1 = q_3 = r_3 = 1$ is globally optimal, the policy is exactly the one that is described by Propositions 4 or 5. In particular, the monotonicity constraint (10) binds, that is, $v^H P_{L|H} = v^L P_{L|L}$, and ($q_2$, r_2) equals to either $\left(\min\{\hat{q}_2, \bar{q}_2\}, \hat{r}_2(q_2)\right)$ or $\left(\underline{q}_2, 0\right)$. The constraint (10) also binds under the optimal disclosure policies of Table 1 (strong heterogeneity and $\alpha \in [0.09, 0.13]$), so that even in this "special" case, the designer sends positively correlated signals to the contestants.[30] At the same time, here we observe $r_1 \leq q_1 < r_3 \leq q_3 < 1$. Thus, the symmetric states are not revealed precisely. Also, state (L, L) is disclosed precisely much more often (r_3 is close to one) than state (H, H)

[30] Here $r_2 = \hat{r}_2(\cdot)$ where $\hat{r}_2(\cdot)$ solves $v^H P_{L|H} = v^L P_{L|L}$.

that generates the highest expected aggregate effort in the perfect information setting. In other words, if heterogeneity in valuations is strong enough, the designer can benefit from persuading both contestants that they are of type L. Importantly, such a policy performs best only for a sufficiently small α when state $(L,\, L)$ is indeed likely. The following result summarizes theses findings.

Result 2. *The optimal signal distribution* $\pi\left(\{q_j^*, r_j^*\}_{j\in 1:3}\right)$ *can be characterized as follows:*

- *If heterogeneity in prize valuations is mild or moderate, the symmetric states* $(H,\, H)$ *and* $(L,\, L)$ *are revealed precisely, that is,* $q_1^* = r_1^* = q_3^* = r_3^* = 1$*, while the asymmetric states* $(H,\, L)$ *and* $(L,\, H)$ *are revealed with a noise:*

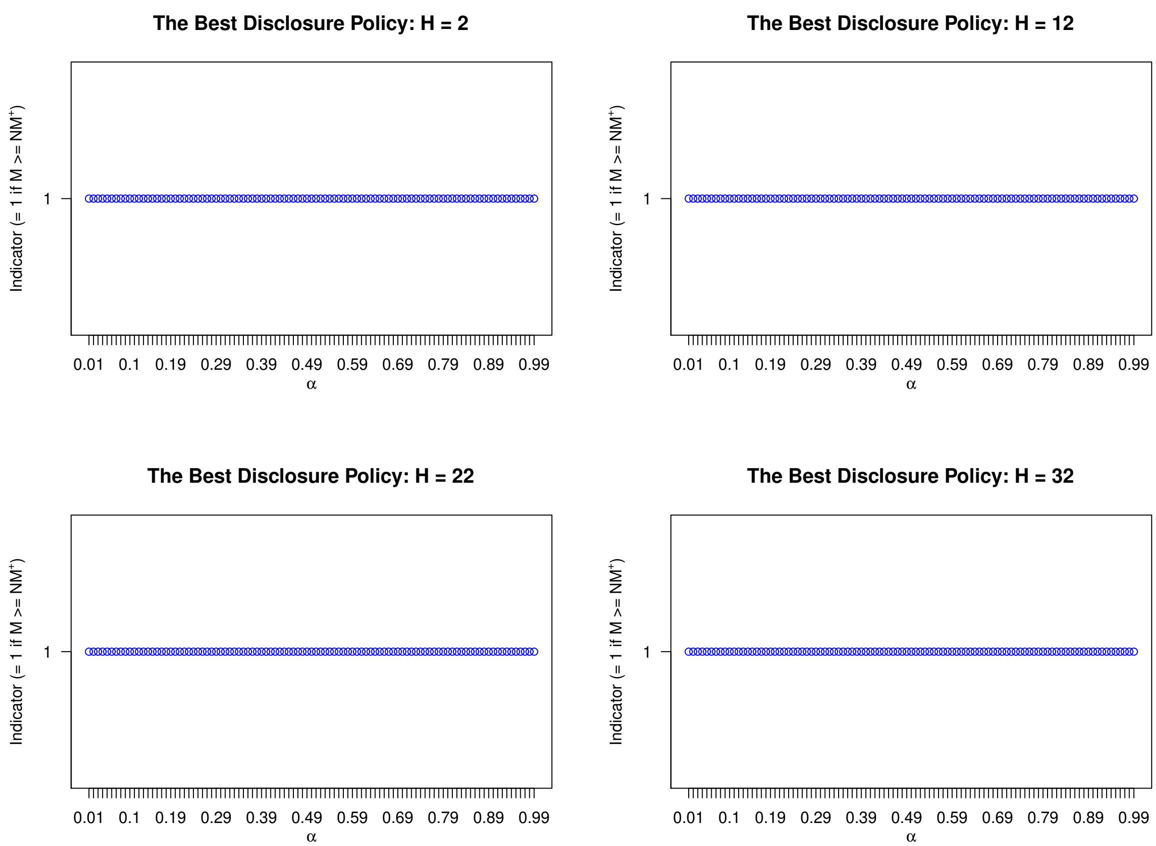
$$q_2^* = \max\left\{\min\left\{\hat{q}_2,\, \bar{q}_2\right\},\, \underline{q}_2 I_{\left\{\underline{q}_2<1\right\}}\right\} \in (0,\, 1) \text{ and } r_2 = \hat{r}_2\left(q_2\right) \in (0,\, 1)$$

  where $\hat{q}_2$, $\bar{q}_2$, $\underline{q}_2$, *and* $\hat{r}_2\left(q_2\right)$ *are defined in Propositions 4 and 5, and* I *is an index function equal to one if the condition in curly brackets holds, and zero otherwise. Thus, the optimal policy coincides with the one in Proposition 4 or the one in Proposition 5.*

- *If heterogeneity in prize valuations is strong, there exist* $\breve{\alpha}_1, \breve{\alpha}_2 > 0$ *such that* $\breve{\alpha}_1 < \breve{\alpha}_2 < \frac{H-3L}{3(H-L)}$ *for which the following observations are true:*
  - *for* $\alpha \in [0,\, \breve{\alpha}_1) \cup (\breve{\alpha}_2,\, 1]$*, the optimal precision parameters are* $q_1^* = r_1^* = q_3^* = r_3^* = 1$, $q_2^* = \max\left\{\hat{q}_2,\, \underline{q}_2\right\} \in (0,\, 1)$*, and* $r_2^* = \hat{r}_2\left(q_2\right) \in [0,\, 1)$*,*
  - *for* $\alpha \in [\breve{\alpha}_1,\, \breve{\alpha}_2]$*, the optimal precision parameters are such that condition* $v^H P_{L|H} = v^L P_{L|L}$ *holds (i.e., monotonicity condition* (10) *binds) and* $r_1^* \le q_1^* < r_3^* \le q_3^* < 1$.

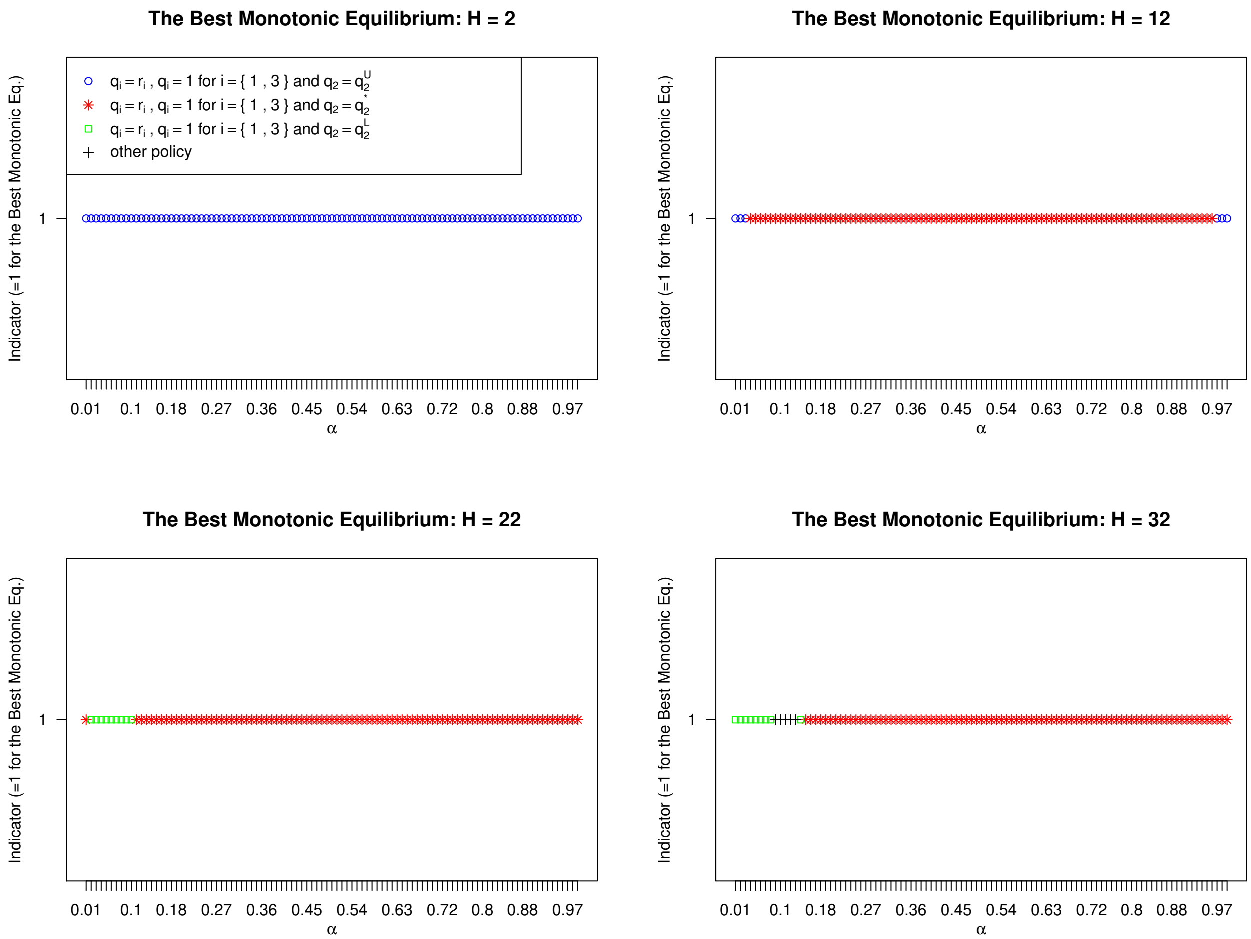
To sum up, our simulations reveal that the optimal disclosure policy always induces the monotonic equilibrium in the contest game. Under this policy, the designer sends private and slightly positively correlated signals. Most often (around 98.7% of cases in our simulations), she chooses to reveal the symmetric states precisely and to signal the asymmetric states with a noise. A deviation from this strategy can be profitable only when heterogeneity in prize valuations is sufficiently strong, while the prior α is weak.

Figure 3: The Best Disclosure Policy as a Function of α and Heterogeneity in Prize Valuations



Note: this figure indicates for which α and H policy M constitutes a global optimum. The indicator equals to 1 if and only if $M \geq NM^{+}$, where M corresponds to the best disclosure rule that supports a monotonic equilibrium and NM^{+} stands for the best rule that induces a non-monotonic equilibrium with strong positive correlation. In all simulations, $L \equiv 1$ and α takes values between 0.01 and 0.99 with a step of 0.01.

Figure 4: The Optimal Disclosure Policy as a Function of α and Heterogeneity in Prize Valuations



Note: this figure indicates for which α and H policy M of a particular form constitutes a global optimum. Here, $\mathrm{q}_2^u = \bar{q}_2$, $\mathrm{q}_2^* = \hat{q}_2$, and $\mathrm{q}_2^{\mathrm{L}} = \underline{q}_2$ where $\bar{q}_2$, $\hat{q}_2$, and $\underline{q}_2$ are defined in Propositions 4 and 5. For other policies, we run a grid search over the signal distribution parameters $(r_1, r_2, r_3, q_1, q_2, q_3)$ that satisfy all the constraints, and choose a scheme with the highest ex ante expected aggregate effort. In all simulations, $L \equiv 1$ and α takes values between 0.01 and 0.99 with a step of 0.01.

Table 1: Policy M that does not feature $q_1 = r_1 = q_3 = r_3 = 1$

α	r_1	r_2	r_3	q_1	q_2	q_3	$\hat{r}_2(\cdot)$	J^*
0.09	0.00	0.01	0.98	0.00	0.42	0.99	0.01	3.986
0.10	0.05	0.00	0.98	0.19	0.42	0.99	0.00	4.331
0.11	0.00	0.01	0.94	0.00	0.34	0.94	0.01	5.050
0.12	0.03	0.00	0.99	0.46	0.41	0.99	0.00	5.022
0.13	0.01	0.10	0.98	0.10	0.48	0.98	0.10	5.370

Note: this table shows policies M that deviate from $q_1 = r_1 = q_3 = r_3 = 1$, and J^* denotes the ex ante expected aggregate effort induced by each policy. These policies are found by running a grid search over the signal distribution parameters (r_1, r_2, r_3, q_1, q_2, q_3) that satisfy constraints (8)–(14), and choosing a scheme with the highest ex ante expected aggregate effort. In all simulations, $H = 32$, and α takes values between 0.09 and 0.13 with a step of 0.01.

7 Conclusion

In this paper we address the question, whether the designer of a contest with unknown prize valuations whose objective is to stimulate as much competitive efforts as possible can gain by choosing the information technology, that is, the precision of the signal about valuations in each state and the way of communicating the signal to the contestants. A combination of three important features distinguishes our model from other studies on the optimal information disclosure/feedback in contests: (i) a broad range of disclosure rules available to the designer, (ii) an assumption that both contestants are initially uninformed about their values of winning, and (iii) the continuous space for the contestants' possible effort choices. Similarly to the literature on the Bayesian persuasion, we also assume that the designer commits to a certain information technology before the signal about valuations is realized. Upon its realization, she reveals the signal to the contestants by means of the announced type of information disclosure: publicly, privately or not at all. Then a static contest game begins, in which the equilibrium effort choices depend on the disclosure policy adopted by the designer.

We find that the highest expected aggregate effort can be obtained if the designer employs private signals that are (slightly) positively correlated (the monotonicity condition binds) and that reveal the true contestants' values of winning precisely if and only if these values are the same. Such signal distribution is proved to be at least one of the local optima under private disclosure when the prior probability of having a high prize valuation, α, is sufficiently high.

Then, using numerical simulations we show that this policy is, in fact, globally optimal in 98.7% of all cases. Public disclosure is never optimal, but when it has to be used, – as may be the case in many real-life contests, – the optimal signal distribution must induce symmetric beliefs. These findings confirm that the best the designer can do to stimulate effort is to make contestants believe that their values of winning, and hence the incentives to compete, are likely to be the same. The perceived asymmetry reduces the intensity of competition, prompting the designer to make signals about asymmetric states noisy.

Research in this paper can be extended in several directions. For example, one could study the implications of adding a possibility of communication between contestants before the start of the competition. There, both the choice of the disclosure policy by the designer and the incentives of the contestants to share information could be explored. Furthermore, in a setting with more than two contestants, one could model the communication structure by a network, in which only the directly linked players can share information with each other. Such local nature of communication introduces the role for the social network structure in the designer's choice of the disclosure policy and in the contestants' choice of whether to share their signal or not. Another interesting possibility is to conduct an experiment so as to investigate how actual subjects in a contest game with unknown prize valuations respond to different disclosure policies. In particular, one could explore whether the theoretically optimal disclosure rule, that involves private and partially precise signals, actually induces a higher aggregate effort than some more straightforward disclosure policies, such as, for instance, complete nondisclosure, where subjects have to rely only on the prior.

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Appendix A: Examples of optimal signal distributions under public disclosure

Example 1. ***Concealment***
Suppose the designer chooses to send signals that are completely uninformative about the state:

$$r_j^{pub} = \frac{1}{4} \text{ and } q_j^{pub} = \frac{1}{2} \text{ for any } j \in 1:4$$

Then, upon receiving any signal $\mathbf{m}$, contestants stay with their (symmetric) prior. It is easy to see that the expected aggregate effort in this case is, indeed, equal to $J^{pub} = \frac{v_1^m}{2} + \frac{v_1^m}{2} = v_i^m = \alpha H + (1-\alpha)$.

Example 2. ***Partially precise information revelation***
Another policy that respects symmetric belief updating is to disclose precisely both symmetric states and convey no information when the contestants' prize valuations turn out to be asymmetric:

$$r_1^{pub} = r_4^{pub} = q_1^{pub} = q_4^{pub} = 1 \text{ and } r_2^{pub} = r_3^{pub} = q_2^{pub} = q_3^{pub} = \frac{1}{2}$$

The expected aggregate effort generated by this policy is

$$\alpha^2 H + 2\alpha\left(1-\alpha\right)\left(\frac{H}{2} + \frac{1}{2}\right) + \left(1-\alpha\right)^2$$

which is exactly equal to $\alpha H + (1-\alpha)$.

Appendix B: Proofs

Proof. [Proof of **Proposition 1**] To characterize the equilibrium of a symmetric contest game with private signals, we use Proposition 1 of Liu and Chen (2016). In our setting

$$v^H \equiv H_{LC},\ v^L \equiv L_{LC}, \text{ and } V_{LC} = 1$$

where the LC subscript corresponds to the original model of Liu and Chen (2016). Then, our equilibrium characterization follows immediately if one substitutes v^H and v^L into Proposition 1 of Liu and Chen (2016).

First, consider a monotonic equilibrium that requires $\frac{v^H}{v^L} \geq \frac{P_{L|L}}{P_{L|H}}$ and $\frac{v^H}{v^L} \geq \frac{P_{H|L}}{P_{H|H}}$. In this equilibrium

- A contestant who gets message L, or type L, randomizes on the $[0, \underline{e}]$ interval with probability 1, where $\underline{e} = P_{L|L}v^L$, and
- A contestant who receives message H, or type H, randomizes on the $[\underline{e}, \bar{e}]$ interval with probability 1, where $\bar{e} = \underline{e} + P_{H|H}v^H$

and the ex ante expected aggregate effort reaches

$$J^M = P_{HH}\left(\underline{e} + \bar{e}\right) + 2P_{HL}\left(\frac{\underline{e}}{2} + \frac{\underline{e} + \bar{e}}{2}\right) + P_{LL}\underline{e} = \underline{e} + \left(P_{HH} + P_{HL}\right)\bar{e} = P_{L|L}v^L + \alpha P_{H|H}v^H$$

The expected equilibrium payoffs the two contestants' types get are equal to

$$u_H^M = P_{H|H}\frac{1}{2}v^H + P_{L|H}v^H - \frac{\underline{e}+\bar{e}}{2} = P_{L|H}v^H - P_{L|L}v^L$$

$$u_L^M = P_{L|L}\frac{1}{2}v^L - \frac{\underline{e}}{2} = 0$$

Second, take a non-monotonic equilibrium with strong positive correlation that arises when $\frac{v^H}{v^L} < \frac{P_{L|L}}{P_{L|H}}$ holds. In this equilibrium

- Type L randomizes on the $[0,\, \underline{e}]$ interval with probability 1, where $\underline{e} = \frac{v^H v^L\left(P_{L|L}-P_{L|H}\right)}{P_{H|H}v^H - P_{H|L}v^L}$, and
- Type H randomizes on the $[0,\, \underline{e}]$ interval with probability $p_B^H = \frac{P_{L|L}v^L - P_{L|H}v^H}{P_{H|H}v^H - P_{H|L}v^L}$ and exhausts the remaining bidding probability on the $[\underline{e},\, \bar{e}]$ interval, where $\bar{e} = v^H$.

The ex ante expected aggregate effort is

$$J^{NM,+} = P_{HH}\left(\underline{e} + \left(1-p_B^H\right)\bar{e}\right) + 2P_{HL}\left(\frac{\underline{e}}{2}\left(1+p_B^H\right) + \left(1-p_B^H\right)\frac{\underline{e}+\bar{e}}{2}\right) + P_{LL}\underline{e} =$$

$$\underline{e} + \bar{e}\left(P_{HH}+P_{HL}\right)\left(1-p_B^H\right) = \frac{v^H v^L\left(P_{L|L}-P_{L|H}\right)}{P_{H|H}v^H - P_{H|L}v^L} + \frac{v^H \alpha\left(v^H - v^L\right)}{P_{H|H}v^H - P_{H|L}v^L} =$$

$$\frac{v^H\left(v^L\left(P_{L|L}-P_{L|H}\right) + \alpha\left(v^H - v^L\right)\right)}{P_{H|H}v^H - P_{H|L}v^L}$$

and the expected equilibrium payoffs amount to

$$u_H^{NM,+} = P_{H|H}\frac{1}{2}v^H + P_{L|H}v^H\left(p_B^H\frac{1}{2} + \left(1-p_B^H\right)\right) - p_B^H\frac{\underline{e}}{2} - \left(1-p_B^H\right)\frac{\underline{e}+\bar{e}}{2} =$$

$$P_{H|H}\frac{1}{2}v^H + P_{L|H}v^H\frac{1}{2} + P_{L|H}v^H\frac{1}{2}\left(1-p_B^H\right) - \frac{\underline{e}}{2} - \left(1-p_B^H\right)\frac{v^H}{2} = \frac{v^H}{2} - P_{H|H}\left(1-p_B^H\right)\frac{v^H}{2} - \frac{\underline{e}}{2} =$$

$$\frac{v^H}{2}\left[1 - \frac{P_{H|H}\left(v^H - v^L\right) + v^L\left(P_{L|L}-P_{L|H}\right)}{P_{H|H}v^H - P_{H|L}v^L}\right] = \frac{v^H}{2}\left[1 - \frac{P_{H|H}v^H - P_{H|L}v^L}{P_{H|H}v^H - P_{H|L}v^L}\right] = 0$$

$$u_L^{NM,+} = P_{L|L}\frac{1}{2}v^L + P_{H|L}p_B^H\frac{1}{2}v^L - \frac{\underline{e}}{2} = \frac{v^L}{2}\left[P_{L|L} + P_{H|L}\frac{P_{L|L}v^L - P_{L|H}v^H}{P_{H|H}v^H - P_{H|L}v^L} - \frac{v^H\left(P_{L|L}-P_{L|H}\right)}{P_{H|H}v^H - P_{H|L}v^L}\right] =$$

$$\frac{v^L v^H}{2\left(P_{H|H}v^H - P_{H|L}v^L\right)}\left[P_{L|L}P_{H|H} - P_{H|L}P_{L|H} - \left(P_{L|L}-P_{L|H}\right)\right] =$$

$$\frac{v^L v^H}{2\left(P_{H|H}v^H - P_{H|L}v^L\right)}\left[P_{L|L}\left(1-P_{L|H}\right) - P_{L|H}\left(1-P_{L|L}\right) - \left(P_{L|L}-P_{L|H}\right)\right] = 0$$

Finally, take a non-monotonic equilibrium with strong negative correlation, that is, $\frac{v^H}{v^L} < \frac{P_{H|L}}{P_{H|H}}$. In this equilibrium

- Type L randomizes on the $[0, \underline{e}]$ interval with probability $p_B^L = \frac{v^H - v^L}{P_{L|H}v^H - P_{L|L}v^L}$ and exhausts the remaining bidding probability on the $[\underline{e}, \bar{e}]$ interval, where $\underline{e} = \frac{P_{L|L}\left(v^H - v^L\right)v^L}{P_{L|H}v^H - P_{L|L}v^L}$ and $\bar{e} = v^L$,
- Type H randomizes on the $[\underline{e}, \bar{e}]$ interval with probability 1.

The ex ante expected aggregate effort achieves

$$J^{NM,-} = P_{HH}\left(\underline{e} + \bar{e}\right) + 2P_{HL}\left(\frac{\underline{e}}{2}p_B^H + \left(1 + 1 - p_B^L\right)\frac{\underline{e} + \bar{e}}{2}\right) + P_{LL}\left(\underline{e} + \left(1 - p_B^L\right)\bar{e}\right) =$$

$$\underline{e} + \bar{e}\left(\alpha + \left(1 - p_B^L\right)(1 - \alpha)\right) = \underline{e} + \bar{e} - \bar{e}p_B^L(1 - \alpha) = v^L + \frac{\left(v^H - v^L\right)v^L\left(P_{L|L} - (1 - \alpha)\right)}{P_{L|H}v^H - P_{L|L}v^L}$$

and the expected equilibrium payoffs are

$$u_H^{NM,-} = P_{H|H}\frac{1}{2}v^H + P_{L|H}v^H\left(p_B^H + \left(1 - p_B^H\right)\frac{1}{2}\right) - \frac{\underline{e} + \bar{e}}{2} =$$

$$P_{H|H}\frac{1}{2}v^H + P_{L|H}v^H\frac{1}{2} + P_{L|H}v^H\frac{p_B^H}{2} - \frac{\underline{e} + \bar{e}}{2} = \frac{v^H}{2}\frac{v^L}{2} + P_{L|H}v^H\frac{p_B^H}{2} - \frac{\underline{e}}{2} =$$

$$\frac{v^H}{2} - \frac{v^L}{2} + \frac{1}{2}\left[\frac{P_{L|H}v^H\left(v^H - v^L\right) - P_{L|L}v^L\left(v^H - v^L\right)}{P_{L|H}v^H - P_{L|L}v^L}\right] = v^H - v^L$$

$$u_L^{NM,-} = P_{L|L}\frac{1}{2}v^L + P_{H|L}\left(1 - p_B^L\right)\frac{1}{2}v^L - p_B^L\frac{\underline{e}}{2} - \left(1 - p_B^L\right)\frac{\underline{e} + \bar{e}}{2} =$$

$$\frac{v^L}{2} - p_B^L P_{H|L}\frac{1}{2}v^L - \frac{\underline{e}}{2} - \left(1 - p_B^L\right)\frac{v^L}{2} = p_B^L P_{L|L}\frac{1}{2}v^L - \frac{\underline{e}}{2} = 0$$

□

Proof. [Proof of **Proposition 2**] Suppose that $v_1^{m,pub} \geq v_2^{m,pub}$ for a subset of signal profiles $\mathbf{m} \in \tilde{M}_\pi \subseteq M$, and $v_1^{m,pub} < v_2^{m,pub}$ for all other $\mathbf{m} \notin \tilde{M}_\pi$. Recall that by definition of $v_i^{m,pub}$, $v_i^{m,pub} \geq v_{-i}^{m,pub}$ if and only if $P\left(v_i = H|\,\mathbf{m}, \pi\right) \geq P\left(v_{-i} = H|\,\mathbf{m}, \pi\right)$. Then, given the equilibrium strategies described in Section 5.1, and in line with equation (1), the ex ante expected aggregate effort of the contestants is equal to

$$J^{pub} \equiv \sum_{\mathbf{m}\in\tilde{M}_\pi} P(\mathbf{m})\frac{v_2^{m,pub}}{2}\left(1 + \frac{v_2^{m,pub}}{v_1^{m,pub}}\right) + \sum_{\mathbf{m}\notin\tilde{M}_\pi} P(\mathbf{m})\frac{v_1^{m,pub}}{2}\left(1 + \frac{v_1^{m,pub}}{v_2^{m,pub}}\right)$$

Thus, the optimization problem of the contest designer in case of publicly revealed signals looks as follows:

$$\max_{\pi\equiv\{q_j, r_j\}_{j=1}^4}\left\{J^{pub} \equiv \sum_{\mathbf{m}\in\tilde{M}_\pi} P(\mathbf{m})\frac{v_2^{m,pub}}{2}\left(1 + \frac{v_2^{m,pub}}{v_1^{m,pub}}\right) + \sum_{\mathbf{m}\notin\tilde{M}_\pi} P(\mathbf{m})\frac{v_1^{m,pub}}{2}\left(1 + \frac{v_1^{m,pub}}{v_2^{m,pub}}\right)\right\} \tag{A-1}$$

$$
\begin{aligned}
\text{s. t. } & P\left(v_1 = H \middle| \mathbf{m}, \pi\right) \geq P\left(v_2 = H \middle| \mathbf{m}, \pi\right) \forall \mathbf{m}_\pi \in \tilde{M}_\pi \\
& P\left(v_2 = H \middle| \mathbf{m}, \pi\right) > P\left(v_1 = H \middle| \mathbf{m}, \pi\right) \forall \mathbf{m}_\pi \notin \tilde{M}_\pi \\
& q_j^{pub} \geq r_j^{pub} \\
& r_j^{pub} \geq \max\left\{0, 2q_j^{pub} - 1\right\} \\
& q_j^{pub}, r_j^{pub} \in [0, 1], j = 1, .., 4
\end{aligned}
$$

Here the first two constraints define the meaning of $\mathbf{m} \in \tilde{M}_\pi$ and $\mathbf{m} \notin \tilde{M}_\pi$, while the remaining inequalities make sure that the signal distribution π is well defined.

The solution of this problem gives an optimal signal distribution π under public disclosure. To find it, we observe that the expression $\frac{v_2^m}{2}\left(1 + \frac{v_2^m}{v_1^m}\right)$ under the first sum in J^{pub} is strictly increasing in v_2^m and decreasing in v_1^m. Then, given the first constraint, which is equivalent to $v_1^m \geq v_2^m$, the expression attains its maximum when $v_1^m = v_2^m$. Similarly, the expression $\frac{v_1^m}{2}\left(1 + \frac{v_1^m}{v_2^m}\right)$ under the second sum in J^{pub} is maximized when the second constraint is as close to equality as possible, which also means $v_1^m = v_2^m$. Condition $v_1^m = v_2^m$ turns the objective function into $\sum_{\mathbf{m} \in M} P(\mathbf{m})\, v_2^m$. Thus, on the set of parameters $\left\{q_j^{pub}, r_j^{pub}\right\}_{j=1}^4$ that satisfy the constraints of the optimization problem in (A-1), the value of the objective function does not exceed $\sum_{\mathbf{m} \in M} P(\mathbf{m})\, v_2^m$. Simple algebra implies that

$$
\sum_{\mathbf{m} \in M} P(\mathbf{m})\, v_2^m = \alpha H + (1 - \alpha)
$$

Hence, we obtain that the upper bound for J^{pub}, given the set of constraints in (A-1), is equal to $\alpha H + (1 - \alpha)L$, and it is achieved at such values of $\left\{q_j^{pub}, r_j^{pub}\right\}_{j=1}^4$ that deliver $v_1^m = v_2^m$ for any signal $\mathbf{m} \in M$. Now, $v_1^m = v_2^m$ (or alternatively, $P\left(v_1 = H \middle| \mathbf{m}\right) = P\left(v_2 = H \middle| \mathbf{m}\right)$) is true for any signal $\mathbf{m} \in M$ if and only if conditions (4) – (6) hold:[31]

$$
\begin{aligned}
q_2^{pub} - r_2^{pub} &= q_3^{pub} - r_3^{pub} \\
r_2^{pub} &= 1 - 2q_3^{pub} + r_3^{pub} \\
r_3^{pub} &= 1 - 2q_2^{pub} + r_2^{pub}
\end{aligned}
$$

Thus, any set of precision parameters $\left\{q_j^{pub}, r_j^{pub}\right\}_{j=1}^4$ that satisfy these conditions delivers an optimal signal distribution π under public disclosure. □

Proof. [Proof of **Proposition 3**] To prove the first statement of the proposition, we construct a private disclosure policy that induces a monotonic equilibrium and generates a higher expected aggregate effort than the best public policy:

- Take $q_1 = r_1 = q_3 = r_3 = 1$ and $q_2 = \frac{1}{2}$,
- Then, the contestants' types look as follows:

[31] This follows immediately from the definition of probabilities $P\left(v_i = H \middle| \mathbf{m}, \pi\right)$ in the beginning of this section and the assumption that $\alpha \in (0, 1)$.

$$v^H = \frac{(1+\alpha)\,H + (1-\alpha)}{2} \equiv v^H(1,\,1)\,,\; v^L = \frac{\alpha H + (2-\alpha)}{2} \equiv v^L(1,\,1)$$

- Assume the monotonicity condition $\left\{v^H P_{L|H} \geq v^L P_{L|L}\right\}$ binds and solve it for r_2:

$$r_2 = \frac{v^L(1,\,1)}{2\left(\alpha v^L(1,\,1) + (1-\alpha)\,v^H(1,\,1)\right)} \equiv \hat{r}_2\left(\frac{1}{2}\right) \in [2q_2 - 1,\, q_2]$$

- With this policy, the expected aggregate effort reaches:

$$J^M\left(1,\,1,\,\tfrac{1}{2},\,\hat{r}_2\left(\tfrac{1}{2}\right),\,1,\,1\right) = (1+\alpha)\left(1 - 2\alpha\hat{r}_2\left(\tfrac{1}{2}\right)\right) v^L(1,\,1) +$$

$$\alpha\left(1 - 2(1-\alpha)\,\hat{r}_2\left(\tfrac{1}{2}\right)\right) v^H(1,\,1)$$

Now, we check if $J^M\left(1,\,1,\,\tfrac{1}{2},\,\hat{r}_2\left(\tfrac{1}{2}\right),\,1,\,1\right)$ outperforms the best public disclosure rule. Notice that $\left\{J^{pub} \equiv \alpha H + (1-\alpha)\right\}$ can be rewritten as follows:

$$J^{pub} \equiv \alpha H + (1-\alpha)\,L = v^L(1,\,1)\,(1-\alpha) + \alpha v^H(1,\,1)$$

The constructed private disclosure rule dominates the optimal public disclosure policy if and only if:

$$(1+\alpha)\left(1 - 2\alpha\hat{r}_2\left(\frac{1}{2}\right)\right) v^L(1,\,1) + \alpha\left(1 - 2(1-\alpha)\,\hat{r}_2\left(\frac{1}{2}\right)\right) v^H(1,\,1) \geq$$

$$v^L(1,\,1)\,(1-\alpha) + \alpha v^H(1,\,1) \Leftrightarrow$$

$$v^L(1,\,1)\left(1 - \hat{r}_2\left(\frac{1}{2}\right)\right) - \hat{r}_2\left(\frac{1}{2}\right)\left(\alpha v^L(1,\,1) + (1-\alpha)\,v^H(1,\,1)\right) \geq 0 \Leftrightarrow$$

$$v^L(1,\,1)\left(\frac{1}{2} - \hat{r}_2\left(\frac{1}{2}\right)\right) \geq 0$$

and the inequality holds for any $v^H(1,\,1) \geq v^L(1,\,1)$. □

Proof. [Proof of **Proposition 4**] Write down the Lagrangian for the designer's optimization program when the monotonic equilibrium is played:

$$L\left(\{q_i,\,r_i\}_{i=1}^3\,,\,\mathscr{M}\right) = J^M\left(\{q_i,\,r_i\}_{i=1}^3\right) + \eta_1\left(v^H P_{L|H} - v^L P_{L|L}\right) +$$

$$\eta_2\left(v^H P_{H|H} - v^L P_{H|L}\right) + \Sigma_{i=1}^3 \lambda_i\,(q_i - r_i) + \Sigma_{i=1}^3 \gamma_i\,(r_i - \max\{0,\,2q_i - 1\}) +$$

$$\chi\left(q_1 - \left(1 - \frac{1-\alpha}{\alpha} q_2\right)\right) + \omega\left(\alpha^2\,(q_1 - 1) + (1-\alpha)^2\,(1 - q_3)\right)$$

where

- $\mathscr{M} = \left\{\{\lambda_j,\,\gamma_j\}_{j=1}^3\,,\,\chi,\,\omega\right\}$ denotes a set of non-negative Lagrange multipliers and
- $J^M\left(\{q_i,\,r_i\}_{i=1}^3\right) \equiv (1+\alpha)\,P_{L|L} v^L + \alpha P_{H|H} v^H$.

Suppose $\left(v^H P_{L|H} = v^L P_{L|L}\right)$ holds in the optimum, which requires:

$$\eta_1 \geq 0,\, \eta_2 = 0$$

Rewrite the monotonicity condition:

$$r_2 = \hat{r}_2\,(q_1,\,r_1,\,q_2,\,q_3,\,r_3) = q_2 - T\,(q_1,\,r_1,\,q_2,\,q_3,\,r_3)$$

where

$$T\left(q_1,\, r_1,\, q_2,\, q_3,\, r_3\right) = \frac{(1-\alpha)\left(v^H - v^L\right)}{2\left((1-\alpha)\, v^H + \alpha v^L\right)} \geq 0 \,\text{for any}\; v^H \geq v^L$$

Then, $\{r_2 = q_2 - T(\cdot)\}$ implies $r_2 \leq q_2$, and $\lambda_2 = 0$ follows.
Next, assume $q_1 = r_1 = q_3 = r_3 = 1$ in the optimum that can be supported with:

$$\lambda_1 \geq 0,\; \lambda_3 \geq 0,\; \gamma_1 \geq 0,\; \gamma_3 \geq 0$$

and verify this later. Now, the $\left\{q_1 \geq \left(1 - \frac{1-\alpha}{\alpha} q_2\right)\right\}$ constraint becomes a strict inequality, and $\chi = 0$ follows. Also, the consistency condition is always satisfied for $q_1 = q_3 = 1$, and we impose $\omega = 0$.

Expressing q_3 as a function of q_1 from the consistency condition and substituting this into the objective function $J(\cdot)$, we get the system of first-order conditions with respect to q_1, r_1, q_2, r_2, and r_3 (recall $\eta_2 = \chi = \omega = 0$ imposed above and $\lambda_2 = 0$ that follows from $\eta_1 \geq 0$):

$$\begin{cases} \frac{\partial L(\cdot)}{\partial r_i} = \frac{\partial J^M(\cdot)}{\partial r_i} + \eta_1 \frac{\partial\left(v^H P_{L|H} - v^L P_{L|L}\right)}{\partial r_i} - \lambda_i + \gamma_i = 0,\; i \in \{1,\, 2,\, 3\} \\ \frac{\partial L(\cdot)}{\partial q_1} = \frac{\partial J^M(\cdot)}{\partial q_1} + \eta_1 \frac{\partial\left(v^H P_{L|H} - v^L P_{L|L}\right)}{\partial q_1} + \lambda_1 - 2\gamma_1 I\left\{q_1 \geq \frac{1}{2}\right\} - 2\frac{\alpha^2}{(1-\alpha)^2}\gamma_3 I\left\{q_3 \geq \frac{1}{2}\right\} = 0 \\ \frac{\partial L(\cdot)}{\partial q_2} = \frac{\partial J^M(\cdot)}{\partial q_2} + \eta_1 \frac{\partial\left(v^H P_{L|H} - v^L P_{L|L}\right)}{\partial q_2} - 2\gamma_2 I\left\{q_2 \geq \frac{1}{2}\right\} = 0 \end{cases}$$

With all the assumptions made, the set of possibly inactive constraints reduces to:

$$\begin{cases} \hat{r}_2(q_2) \geq 0 & (1.1) \\ \hat{r}_2(q_2) \geq 2q_2 - 1 & (1.2) \end{cases}$$

where $\hat{r}_2(q_2) \equiv \hat{r}_2(1,\, 1,\, q_2,\, 1,\, 1)$. Consider condition (1.1) in detail. The underlying equation has two real roots with respect to q_2:

$$q_2 = 0 \,\text{and}\, q_2 = \begin{cases} \frac{H-3-3\alpha(H-1)}{2(1-2\alpha)(H-1)} \equiv \underline{q}_2, & \alpha \neq \frac{1}{2} \\ 0, & \alpha = \frac{1}{2} \end{cases}$$

The only case when $\underline{q}_2$ turns to be feasible corresponds to $H > 3$ and $\alpha \in \left[0,\, \frac{H-3}{3(H-1)}\right]$. Otherwise, it is either (1) $\underline{q}_2 < 0$ and $\hat{r}_2(q_2) \geq 0$ for any $q_2 \geq \max\left\{0,\, \underline{q}_2\right\}$ or (2) $\underline{q}_2 > 1$ and $\hat{r}_2(q_2) \geq 0$ for any $q_2 \leq \min\left\{1,\, \underline{q}_2\right\}$. Thus, we define the lower bound on q_2:

- $H > 3$ and $\alpha \in \left[0,\, \frac{H-3}{3(H-1)}\right] \Rightarrow$ it must be $q_2 \geq \underline{q}_2$,
- Otherwise, $q_2 \geq 0$.

Thus, fixing $\alpha \geq \max\left\{0,\, \frac{H-3}{3(H-1)}\right\}$ ensures that the lower bound on q_2 is zero.

Further, take inequality (1.2) and find the roots of $\{\hat{r}_2(q_2) \geq 2q_2 - 1\}$ with respect to q_2:

$$q_2 = \begin{cases} \frac{H-3-5\alpha(H-1)\pm\sqrt{D}}{4(H-1)(1-2\alpha)} \equiv q_2^{1,\,2}, & \alpha \neq \frac{1}{2} \\ \frac{2(H+1)}{3H+1}, & \alpha = \frac{1}{2} \end{cases}$$

where

$$D = -7(H-1)\,\alpha^2 + 6\left(H^2 - 4H + 3\right)\alpha + H^2 + 10H - 7 > 0\,\forall\, \alpha \in [0,\, 1]$$

For $\alpha = \frac{1}{2}$, there is only one root with respect to q_2, and $\{\hat{r}_2(q_2) \geq 2q_2 - 1\}$ holds for any $q_2 \leq \frac{2(H+1)}{3H+1}$. Next, take the case of $\alpha \neq \frac{1}{2}$. Here, q_2^1 belongs to the $[0,\, 1]$ interval for any $\alpha \in [0,\, 1]$. At the same time, q_2^2 turns to be negative for $\alpha \in \left[0,\, \frac{1}{2}\right)$ and exceeds 1 for $\alpha \in \left(\frac{1}{2},\, 1\right]$. Hence, q_2^2 is never feasible. Taking the feasibility constraint into account, the $\{\hat{r}_2(q_2) \geq 2q_2 - 1\}$ inequality holds if and only if $q_2 \in [0,\, q_2^1]$. Thus, we have an upper bound on q_2:

$$q_2 \leq \bar{q}_2 = \begin{cases} q_2^1, & \alpha \neq \frac{1}{2} \\ \frac{2(H+1)}{3H+1}, & \alpha = \frac{1}{2} \end{cases}$$

First, consider the case when the $q_2 \in [0,\, \bar{q}_2]$ constraint does not bind in the optimum (namely, $\gamma_2 = 0$ holds). The system to solve for q_2 and r_2 reduces to:

$$\begin{cases} \eta_1 = -\frac{\partial J^M(\cdot)}{\partial r_2} / \frac{\partial\left(v^H P_{L|H} - v^L P_{L|L}\right)}{\partial r_2} = -\frac{\partial J^M(\cdot)}{\partial q_2} / \frac{\partial\left(v^H P_{L|H} - v^L P_{L|L}\right)}{\partial q_2} \\ r_2 = \hat{r}_2(q_2) \end{cases}$$

where the first equation comes from $\left(\frac{\partial L(\cdot)}{\partial q_2} = \frac{\partial L(\cdot)}{\partial r_2} = 0\right)$, $\eta_1 \geq 0$, and $\gamma_2 = 0$. This delivers two real roots for q_2:[32]

$$q_2 = \begin{cases} \frac{\alpha \pm \sqrt{\alpha(1-\alpha)}(1+\alpha(H-1))}{\alpha(2\alpha-1)(H-1)} \equiv \hat{q}_2^{1,\,2}, & \alpha \neq \frac{1}{2} \\ \frac{H+1}{2(H-1)}, & \alpha = \frac{1}{2} \end{cases}$$

where

$$\hat{q}_2^1 < 0 \,\forall\, \alpha \in \left[0,\, \frac{1}{2}\right) \text{ and } \hat{q}_2^2 > 1 \,\forall\, \alpha \in \left(\frac{1}{2},\, 1\right]$$

$$\hat{q}_2^2 > 0 \,\forall\, \alpha \in [0,\, 1] \text{ and}$$

$$\hat{q}_2^2 \leq 1 \Leftrightarrow (H \geq 3) \text{ and } \alpha \in [\alpha_1,\, \alpha_2] \text{ where}$$

$$\alpha_{1,\,2} = \frac{H - 1 \mp \sqrt{(H-3)(H+1)}}{2(H-1)}$$

$$\frac{H+1}{2(H-1)} \leq 1 \Leftrightarrow H \geq 3$$

When $\alpha \neq \frac{1}{2}$, the only root that can be feasible under certain conditions, is q_2^2, and we define the ultimate solution – $\hat{q}_2$ – as follows:

$$\hat{q}_2 = \begin{cases} \frac{\alpha - \sqrt{\alpha(1-\alpha)}(1+\alpha(H-1))}{\alpha(2\alpha-1)(H-1)}, & \alpha \neq \frac{1}{2} \\ \frac{H+1}{2(H-1)}, & \alpha = \frac{1}{2} \end{cases}$$

One must verify when $\hat{q}_2 \in (0,\, \bar{q}_2)$ holds and, hence, supports $\gamma_2 = 0$.[33] With $\alpha \geq \max\left\{0,\, \frac{H-3L}{3(H-L)}\right\}$, the $\{\hat{q}_2 > 0\}$ inequality is always satisfied. Next, we solve $\{\hat{q}_2 < \bar{q}_2\}$ for $\alpha \neq \frac{1}{2}$. The corresponding equation has two roots:

[32]All calculations were performed with Matlab symbolic toolbox.

[33]Recall that $\{\hat{r}_2(\hat{q}_2) < \hat{q}_2\}$ always holds when the monotonicity condition binds, and $\lambda_2 = 0$ follows.

$$\hat{\alpha}_{1,2} = \frac{H - 1 \mp \sqrt{(H+3)(H-5)}}{2(H-1)}$$

When $H \in (1, 5)$, these roots are complex, and $\{\hat{q}_2 > \bar{q}_2\}$ for any α. If $H \geq 5$, both $\hat{\alpha}_1$ and $\hat{\alpha}_2$ become real, and $\hat{\alpha}_{1,2} \in (0, 1)$ holds where $\hat{\alpha}_1 > \alpha_1$ and $\hat{\alpha}_2 < \alpha_2$ for any $H > 1$. Then, $\alpha \in (\hat{\alpha}_1, \hat{\alpha}_2)$ supports $\{\hat{q}_2 < \bar{q}_2\}$. For $\alpha \neq \frac{1}{2}$, the $\{\hat{q}_2 < \bar{q}_2\}$ inequality holds if and only if $H > 5$. Putting things together, we get that:

- For $H \in (1, 5)$, $\hat{q}_2$ is not feasible (namely, $\hat{q}_2 > \bar{q}_2$), and the $\{q_2 \leq \bar{q}_2\}$ constraint must bind implying $\gamma_2 \geq 0$ in the optimum,
- For $H \geq 5$, the $\{\hat{q}_2 < \bar{q}_2\}$ constraint always holds for $\alpha = \frac{1}{2}$ and requires $\alpha \in (\hat{\alpha}_1, \hat{\alpha}_2)$ for $\alpha \neq \frac{1}{2}$.

Here, $\hat{q}_2$ is the only stationary point in the $q_2 \in [0, \bar{q}_2]$ interval. Then, to illustrate that $q_2 = \hat{q}_2$ is a maximum, we compare the value of the objective function $J^M(\cdot)$ at $q_2 = 0$, $q_2 = \bar{q}_2$, and $q_2 = \hat{q}_2$, respectively:

- $J^M(\cdot, \hat{r}_2(0), 0) < J^M(\cdot, \hat{r}_2(\hat{q}_2), \hat{q}_2)$ holds for any $\alpha \in [0, 1]$ and
- $J^M(\cdot, \hat{r}_2(\bar{q}_2), \bar{q}_2) < J^M(\cdot, \hat{r}_2(\hat{q}_2), \hat{q}_2)$ is true if and only if $\alpha \in (\hat{\alpha}_1, \hat{\alpha}_2)$.

Hence, if $\hat{q}_2$ is feasible, it constitutes a maximum. Otherwise, $q_2 = \bar{q}_2$ must be chosen:

$$J^M(\cdot, \hat{r}_2(0), 0) < J^M(\cdot, \hat{r}_2(\bar{q}_2), \bar{q}_2) \ \forall\, \alpha\, [0, 1]$$

Finally, we restore the set of Lagrange multipliers that support the optimality of $q_1 = r_1 = q_3 = r_3 = 1$ and $\eta_1 \geq 0$ for $\alpha \geq \max\left\{0, \frac{H-3}{3(H-1)}\right\}$. First, take the case when $\hat{q}_2$ is feasible, i.e. $\gamma_2 = 0$. Then, the following set of Lagrange multipliers solves the system of first-order condition:

$$\eta_1 = \begin{cases} \frac{3\alpha^2 - \alpha - \alpha\sqrt{\alpha(1-\alpha)}}{2\alpha - 1}, & \alpha \neq \frac{1}{2} \\ \frac{3}{4}, & \alpha = \frac{1}{2} \end{cases}$$

$$\lambda_i = \gamma_i = 0,\ i \in \{1, 2, 3\}$$

where $\eta_1 > 0$ for any $\alpha \in [0, 1]$. Second, we solve for the multipliers that support $\{\hat{q}_2 > \bar{q}_2\}$ as the optimum:

$$\eta_1 = \left(-\frac{\partial J^M(\cdot)}{\partial r_2} - \frac{1}{2}\frac{\partial J^M(\cdot)}{\partial q_2}\right) \Big/ \left(\frac{\partial\left(v^H P_{L|H} - v^L P_{L|L}\right)}{\partial r_2} + \frac{1}{2}\frac{\partial\left(v^H P_{L|H} - v^L P_{L|L}\right)}{\partial q_2}\right)|_{(\bar{q}_2, \hat{r}_2(\bar{q}_2))}$$

$$\gamma_2 = \left(-\frac{\partial J^M(\cdot)}{\partial r_2} - \eta_1 \frac{\partial\left(v^H P_{L|H} - v^L P_{L|L}\right)}{\partial r_2}\right)|_{(\bar{q}_2, \hat{r}_2(\bar{q}_2))},\ \lambda_2 = 0$$

$$\gamma_3 = \begin{cases} \left(-\frac{\partial J^M(\cdot)}{\partial r_3} - \eta_1 \frac{\partial\left(v^H P_{L|H} - v^L P_{L|L}\right)}{\partial r_3}\right)|_{(\bar{q}_2, \hat{r}_2(\bar{q}_2))}, & \left(\frac{\partial J^M(\cdot)}{\partial r_3} + \eta_1 \frac{\partial\left(v^H P_{L|H} - v^L P_{L|L}\right)}{\partial r_3}\right)|_{(\bar{q}_2, \hat{r}_2(\bar{q}_2))} < 0 \\ 0, & \left(\frac{\partial J^M(\cdot)}{\partial r_3} + \eta_1 \frac{\partial\left(v^H P_{L|H} - v^L P_{L|L}\right)}{\partial r_3}\right)|_{(\bar{q}_2, r_2(\bar{q}_2))} \geq 0 \end{cases}$$

$$\lambda_1 = 2\gamma_1 + 2\gamma_3 \frac{\alpha^2}{(1-\alpha)^2} = \left(\frac{\partial J^M(\cdot)}{\partial r_1} + \eta_1 \frac{\partial\left(v^H P_{L|H} - v^L P_{L|L}\right)}{\partial r_1}\right)|_{(\bar{q}_2, \hat{r}_2(\bar{q}_2))}$$

$$\lambda_3 = \left(\frac{\partial J^M(\cdot)}{\partial r_3} + \eta_1 \frac{\partial \left(v^H P_{L|H} - v^L P_{L|L} \right)}{\partial r_3} + \gamma_3 \right) |_{(\bar{q}_2,\, \hat{r}_2(\bar{q}_2))}$$

To ensure that all the multipliers are non-negative, it is sufficient to verify when $\eta_1 \geq 0$, $\gamma_2 \geq 0$, and $\left\{ \frac{\partial J^M(\cdot)}{\partial r_1} + \eta_1 \frac{\partial \left(v^H P_{L|H} - v^L P_{L|L} \right)}{\partial r_1} \geq 0 \right\}$ hold:

$$\eta_1 \geq 0 \, \forall \, \alpha \in [0,\, 1]$$

$$\gamma_2 \geq 0 \Leftrightarrow \begin{cases} H < 5 \\ \alpha \in [0,\, 1] \end{cases} \text{or} \begin{cases} H \geq 5 \\ \alpha \in [0,\, \hat{\alpha}_1] \cup [\hat{\alpha}_2,\, 1] \end{cases}$$

$$\left(\frac{\partial J^M(\cdot)}{\partial \partial_1} + \eta_1 \frac{\partial \left(v^H P_{L|H} - v^L P_{L|L} \right)}{\partial r_1} \right) |_{(\bar{q}_2,\, \hat{r}_2(\bar{q}_2))} \geq 0 \Leftrightarrow \begin{cases} H < 5 \\ \alpha \in [0,\, 1] \end{cases} \text{or} \begin{cases} H \geq 5 \\ \alpha \in [0,\, \hat{\alpha}_1] \cup [\hat{\alpha}_2,\, 1] \end{cases}$$

where $\hat{\alpha}_1$ and $\hat{\alpha}_2$ were introduced above. Thus, if $\hat{q}_2$ is not feasible, the Lagrange multipliers that support the optimality of $q_1 = r_1 = q_3 = r_3 = 1$, $q_2 = \bar{q}_2$, and $r_2 = \hat{r}_2(\bar{q}_2)$ are well defined. □

Proof. [Proof of **Proposition 5**] To support the claim, we refer to the proof of Proposition 4. Fix $H \geq 3$ and $\alpha \in \left[0,\, \frac{H-3}{3(H-1)}\right]$. For $q_1 = r_1 = q_3 = r_3 = 1$ and the binding monotonicity constraint, we can rewrite the original optimization program in term of q_2:[34]

$$\max_{q_2} \left\{ (1+\alpha) P_{L|L} v^L + \alpha P_{H|H} v^H \right\}$$

$$s.t.\, q_2 \in \left[\underline{q}_2,\, \bar{q}_2 \right] \neq \emptyset$$

$$\underline{q}_2 = \frac{H - 3 - 3\alpha (H-1)}{2(1-2\alpha)(H-1)}$$

where $H \geq 3$ and $\alpha \in \left[0,\, \frac{H-3}{3(H-1)}\right]$ ensure that the lower bound on q_2 is positive, and $\underline{q_2}$ solves $\hat{r}_2(q_2) = 0$. As we showed in the proof of Proposition 4, the interior solution of the program for $\alpha \neq \frac{1}{2}$ corresponds to:

$$\hat{q}_2 = \frac{\alpha - \sqrt{\alpha(1-\alpha)}\,(1 + \alpha(H-1))}{\alpha(2\alpha - 1)(H-1)}$$

First, we check if there exist α's such that $\left\{ \hat{q}_2 < \underline{q}_2 \right\}$ holds. The underlying equation solved for α has two roots:

$$\tilde{\alpha}_{1,\,2} = \frac{H - 9 \mp \sqrt{H^2 - 18H + 1}}{10(H-1)}$$

For $H \in \left(1,\, \left(9 + 4\sqrt{5}\right)\right)$, both roots are complex, and $\left\{ \hat{q}_2 > \underline{q}_2 \right\}$ for any feasible α. When $H \geq \left(9 + 4\sqrt{5}\right)$, $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ become feasible, and $\alpha \in (\tilde{\alpha}_1,\, \tilde{\alpha}_2)$ supports $\left\{ \hat{q}_2 < \underline{q}_2 \right\}$. As $\hat{q}_2$ is always positive, $\alpha \in (\tilde{\alpha}_1,\, \tilde{\alpha}_2)$ implies $\alpha \in \left[0,\, \frac{H-3}{3(H-1)}\right]$ for $H \geq \left(9 + 4\sqrt{5}\right)$.

[34] See the proof of Proposition 4 for more detail.

Second, we verify if the value of the objective function $J^m(\cdot)$ at the lower bound of q_2 is greater than $J^M(\cdot)$ at the upper bound when $\left\{\hat{q}_2 < \underline{q}_2\right\}$ holds. Here, we solve the next condition for α:[35]

$$J^M\left(\cdot,\, \underline{q}_2,\, 0\right) > J^M\left(\cdot,\, \bar{q}_2,\, \hat{r}_2\left(\bar{q}_2\right)\right)$$

The underlying equation has four roots – $\bar{\alpha}_1$, $\bar{\alpha}_2$, $\bar{\alpha}_3$, and $\bar{\alpha}_4$ – with the following properties:

1. $\bar{\alpha}_1 = -\frac{2}{H-1} < 0$,
2. $\bar{\alpha}_2 > \tilde{\alpha}_2$ for any $H > 1$,
3. $\bar{\alpha}_3 < 0$ for any $H > 1$, and
4. $0 < \bar{\alpha}_4 < \tilde{\alpha}_1$ for any $H > 1$.

Thus, $\left\{J^M\left(\cdot,\, \underline{q}_2,\, 0\right) - J^M\left(\cdot,\, \bar{q}_2,\, \hat{r}_2\left(\bar{q}_2\right)\right)\right\}$ does not change its sign in the interval where $\left\{\hat{q}_2 < \underline{q}_2\right\}$ is satisfied. Finally, we take the value of $H = 22$ that does not violate the $\left\{H \geq \left(9 + 4\sqrt{5}\right)\right\}$ constraint and show that the sign is positive. Hence, $\left\{J^M\left(\cdot,\, \underline{q}_2,\, 0\right) > J^M\left(\cdot,\, \bar{q}_2,\, \hat{r}_2\left(\bar{q}_2\right)\right)\right\}$ holds when the interior solution $\hat{q}_2$ is located to the left of $\underline{q}_2$. $\square$

[35] All calculations were performed with Matlab symbolic toolbox.