Доказательство «- » в тесте Хаусмана, учебник Maddala

12.10 Hausman's Specification Error Test

Hausman's specification error test[45] is a general and widely used test for the hypothesis of no misspecification in the model.

Let H_0 denote the null hypothesis that there is no misspecification and let H_1 denote the alternative hypothesis that there is a misspecification (of a particular type). For instance, if we consider the regression model

$$y = \beta x + u \tag{12.20}$$

in order to use the OLS procedure, we specify that x is independent of u. Thus the null and alternative hypotheses are:

H_0 : x and u are independent

H_1 : x and u are not independent

To implement Hausman's test, we have to construct two estimators $\hat{\beta}_0$ and $\hat{\beta}_1$, which have the following properties:

$\hat{\beta}_0$ is consistent and efficient under H_0 but is not consistent under H_1.
$\hat{\beta}_1$ is consistent under *both* H_0 and H_1 but is not efficient under H_0.

Then we consider the difference $\hat{q} = \hat{\beta}_1 - \hat{\beta}_0$. Hausman first shows that

$$\text{var}(\hat{q}) = V_1 - V_0$$

where $V_1 = \text{var}(\hat{\beta}_1)$ and $V_0 = \text{var}(\hat{\beta}_0)$, both variances being computed under H_0. Let $\hat{V}(\hat{q})$ be a consistent estimate of $\text{var}(\hat{q})$. Then we use

$$m = \frac{\hat{q}^2}{\hat{V}(\hat{q})}$$

as a χ^2-distribution with d.f. 1 to test H_0 against H_1. This is an asymptotic test.

We have considered only a single parameter β. In the general case where β is a vector of k parameters, V_1 and V_0 will be matrices, $\hat{\beta}_1$, $\hat{\beta}_0$, and $\hat{q}$ will all be vectors, and the Hausman test statistic is

$$m = \hat{\mathbf{q}}'[\hat{\mathbf{V}}(\hat{\mathbf{q}})]^{-1}\hat{\mathbf{q}}$$

which has (asymptotically) a χ^2-distribution with d.f. k.

Since a consideration of the k-parameter case involves vectors and matrices, we discuss the single-parameter case. The derivations in the k-parameter case are all similar.

To prove the result $\text{var}(\hat{q}) = V_1 - V_0$, we first have to prove the result that

$$\text{cov}(\hat{\beta}_0, \hat{q}) = 0$$

The proof proceeds as follows. Under H_0, both $\hat{\beta}_0$ and $\hat{\beta}_1$ are consistent estimates for β. Hence we get

$$\text{plim } \hat{q} = \text{plim } \hat{\beta}_1 - \text{plim } \hat{\beta}_0 = \beta - \beta = 0$$

Consider a new estimator for β defined by

$$\hat{d} = \hat{\beta}_0 + \lambda\hat{q}$$

where λ is any constant. Then $\text{plim } \hat{d} = \beta$. Thus $\hat{d}$ is a consistent estimator of β for all values of λ.

$$V(\hat{d}) = V_0 + \lambda^2 \text{ var}(\hat{q}) + 2\lambda \text{ cov}(\hat{\beta}_0, \hat{q}) \geq V_0$$

Since $\hat{\beta}_0$ is efficient. Thus

$$\lambda^2 \text{ var}(\hat{q}) + 2\lambda \text{ cov}(\hat{\beta}_0, \hat{q}) \geq 0 \qquad (12.21)$$

for all values of λ. We will show that the relationship (12.21) can be satisfied for all values of λ only if $\text{cov}(\hat{\beta}_0, \hat{q}) = 0$.

Suppose that $\mathrm{cov}(\hat{\beta}_0, \hat{q}) > 0$. Then by choosing λ negative and equal to $-\mathrm{cov}(\hat{\beta}_0, \hat{q})/\mathrm{var}(\hat{q})$, we can show that the relationship (12.21) is violated. Thus $\mathrm{cov}(\hat{\beta}_0, \hat{q})$ is not greater than zero.

Similarly, suppose that $\mathrm{cov}(\hat{\beta}_0, \hat{q}) < 0$. Then by choosing λ positive and equal to $-\mathrm{cov}(\hat{\beta}_0, \hat{q})/\mathrm{var}(\hat{q})$, we show that the relationship (12.21) is violated. Thus $\mathrm{cov}(\hat{\beta}_0, \hat{q})$ cannot be greater than or less than zero. Hence we get $\mathrm{cov}(\hat{\beta}_0, \hat{q}) = 0$.

Now since $\hat{\beta}_1 = \hat{\beta}_0 + \hat{q}$ and $\mathrm{cov}(\hat{\beta}_0, \hat{q}) = 0$, we get

$$\mathrm{var}(\hat{\beta}_1) = \mathrm{var}(\hat{\beta}_0) + \mathrm{var}(\hat{q})$$

or

$$\mathrm{var}(\hat{q}) = \mathrm{var}(\hat{\beta}_1) - \mathrm{var}(\hat{\beta}_0) = V_1 - V_0$$

which is the result on which Hausman's test is based.