Algebraic semantics for intuitionistic predicate logics

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Plan

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Intuitionistic propositional logic

We say that a set of formulas L is closed under the rule **modus ponens** (MP) :

$$\frac{A, \quad A \supset B}{B}$$

if $B \in L$ whenever $A \in L$.

The **intuitionistic logic** is the smallest set of intuitionistic formulas closed under substitutions and modus ponens, and containing the following axioms:

(Ax1) $p \supset (q \supset p)$;

(Ax2) $p \supset (q \supset r) \supset ((p \supset q) \supset (p \supset r))$;

(Ax3) $p \wedge q \supset p$;

(Ax4) $p \wedge q \supset q$;

(Ax5) $p \supset (q \supset (p \wedge q))$;

(Ax6) $p \supset p \vee q$;

(Ax7) $q \supset p \vee q$;

(Ax8) $(p \supset r) \supset ((q \supset r) \supset (p \vee q) \supset r)$;

(Ax9) $\bot \supset p$.

The intuitionistic (or Heyting) propositional logic is denoted by **H**

Predicate formulas

Let $\mathrm{Var} = \{v_1, v_2, \ldots\}$ and $PL^N = \{P_i^n \mid i \geq 0\}\,(n \geq 0)$ be a fixed disjoint countable sets. Var elements are called variables and PL^n elements are called ***n*-ary predicate letters**. An **atomic formula without equality** is either $\perp$ or P_i^0, or $P_i^n(x_1, \ldots, x_n)$ for some $n \geq 0$, $x_1, \ldots, x_n \in \mathrm{Var}$. An **atomic formula with equality** can also be of the form $a = b$, where $a, b \in \mathrm{Var}$ and ' $=$ ' is an extra binary predicate letter.

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Intuitionistic predicate formulas (with or without equality) are built from atomic formulas using the propositional connectives $\wedge, \vee, \supset$ and the quantifiers $\forall, \exists$. The abbreviations $\neg A, \top, A \equiv B$ have the same meaning as in the propositional case; $x \neq y$ abbreviates $\neg(x = y)$. For a formula A, a list of variables $\boldsymbol{x} = x_1 \ldots x_n$ and a quantifier $\mathcal{Q} \in \{\forall, \exists\}$, $\mathcal{Q}\boldsymbol{x}A$ denotes $\mathcal{Q}x_1 \ldots \mathcal{Q}x_n A$.

AF, **IF** denote the sets of atomic and intuitionistic formulas without equality respectively. Corresponding sets of formulas with equality are denoted by $\mathbf{AF}^{(=)}, \mathbf{IF}^{(=)}$.

Consider the following predicate axioms for arbitrary x, y and fixed P, q :

(Ax10) $\forall x P(x) \supset P(y)$;

(Ax11) $P(y) \supset \exists x P(x)$;

(Ax12) $\forall x(q \supset P(x)) \supset (q \supset \forall x P(x))$;

(Ax13) $\forall x(P(x) \supset q) \supset (\exists x P(x) \supset q)$.

And also consider the following axioms of equality for arbitrary x, y and fixed P:

(Ax14) $x = x$;

(Ax15) $(x = y) \supset (P(x) \supset P(y))$.

Predicate logics

Definition

A ***superintuitionistic predicate logic (s.p.l.)*** *is a set* $L \subseteq IF$ *such that:*

- *(s1) L contains the axioms of Heyting's propositional calculus* **H***;*
- *(s2) L contains the predicate axioms (Ax10) (Ax13);*
- *(s3) L is closed under the rules*

$$\frac{A, A \supset B}{B} \textit{ modus ponens} \qquad \frac{\forall x A;}{A} \ \forall\textit{-introduction}$$

- *(s4) L is closed under IF-substitutions.*

Definition

A ***superintuitionistic predicate logic with equality (s.p.l.=)*** *is a set* $L \subseteq IF^{=}$*satisfying* $(s1) + (s_4)$ *and:*

- *(s5) L is closed under* $IF^{=}$*-substitutions;*
- *(s6) L contains the axioms of equality (Ax14), (Ax15).*

Algebraic Kripke sheaf semantics [N.-Y. Suzuki, 1999]

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Definition

Let $\boldsymbol{M} = \langle M, \preccurlyeq \rangle$ be a partially ordered set with the least element $0^M \in \boldsymbol{M}$ and D be a contravariant functor from $\boldsymbol{M}$ to SET, which means that:

- *for every $a \in M$ the set $D(a)$ is non-empty;*
- *for every $a, b \in M$ with $a \preccurlyeq b$ there exists a map $D_{ab} : D(b) \to D(a)$;*
- *$D_{aa} = id_{D(a)}$ for every $a \in M$;*
- *$D_{bc} \circ D_{ab} = D_{ac}$ for every $a, b, c \in M$ with $a \preccurlyeq b \preccurlyeq c$.*

*The functor D is called a **domain-sheaf** over $\boldsymbol{M}$, a tuple $\langle M, \preccurlyeq, 0 \rangle$ is called a **Kripke base** and a pair $\langle \boldsymbol{M}, D \rangle$ is called a **Kripke sheaf**.*

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For each $a, b \in M$ and $d \in D(a)$, element $D_{ab}(d)$ is said to be **inheritor** of an element d at the point b. For each sentence $A \in IS_{D(a)}$ and each element $b \in M$ such that $b \succcurlyeq a$ we define inheriting sentence A_{ab} obtained by replacing every occurence of every $u \in D(a)$ by its inheritor $v = D_{ab}(u)$ at $D(b)$.

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Definition

Let $\mathcal{H}$ denote a category of non-degenerate complete Heyting algebras and complete monomorphisms. A contravariant functor P from a Kripke base $M = \langle M, \preccurlyeq \rangle$ to $\mathcal{H}$ is called a Heyting-valued-sheaf over M and $\mathcal{K} = \langle \boldsymbol{M}, D, P \rangle$ is called an ***algebraic Kripke sheaf****.*

Algebraic Kripke sheaf semantics

Definition

A map V which assigns to each pair (a, A) of an element $a \in M$ and an atomic sentence $A \in AF_D$ an element of $P(a)$ is said to be a valuation on $\langle \boldsymbol{M}, D, P \rangle$ if for every $a, b \in \boldsymbol{M}$ with $a \preccurlyeq b$ implies $V(a, A) \leq P_{ab}\left(V\left(b, A_{ab}\right)\right)$

We extend V to a map which to all senteces A of PL^n inductively as follows:

- $V(a, A \wedge B) = V(a, A) \cap^{P(a)} V(a, B)$;
- $V(a, A \vee B) = V(a, A) \cup^{P(a)} V(a, B)$;
- $V(a, A \supset B) = \bigcap_{b:a \leq b}^{P(a)} P_{ab}\left(V\left(b, A_{ab}\right) \rightarrow^{P(b)} V\left(b, B_{ab}\right)\right)$;
- $V(a, \neg A) = \bigcap_{b:a \leq b}^{P(a)} P_{ab}\left(V\left(b, A_{ab}\right) \rightarrow^{P(b)} \mathbf{0}^{P(b)}\right)$;
- $V(a, \forall x A(x)) = \bigcap_{b:a \leq b}^{P(a)} \bigcap_{v \in D(a)}^{P(a)} P_{ab}\left(V\left(b, A_{ab}(v)\right)\right)$;
- $V(a, \exists x A(x)) = \bigcup_{v \in D(a)}^{P(a)} V(a, A(v))$.

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Definition

A pair $\langle \mathcal{K}, V \rangle$ is called an algebraic Kripke sheaf model.

Definition

A formula A is said to be true in an algebraic Kripke sheaf model $\langle \mathcal{K}, V \rangle$ if $V\left(0^{\boldsymbol{M}}, \bar{A}\right) = 1$, this fact is denoted as $\langle \mathcal{K}, V \rangle \models A$. A formula A is said to be valid in an algebraic Kripke sheaf $\mathcal{K}$ if it is true for every valuation V on $\mathcal{K}$.

Heyting-valued structure semantics [A. Dragalin, 1973]

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Let Ω be a complete Heyting algebra, D a set, and $E : D^2 \to \Omega$ a map such that for any $a, b, c \in D$:

E(1) $E(a, b) = E(b, a)$;

E(2) $E(a, b) \wedge E(b, c) \leq E(a, c)$;

E(3) $\bigvee_{a \in D} E(a, a) = 1$.

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Definition

A triple $\langle \Omega, D, E \rangle$ is called an Ω-valued structure or a ***Heyting valued structure*** *(H.v.s) over Ω. A set D is called an individiual domain and its elements are called individuals. A map $E : D^2 \to \Omega$ is called a measure of equality.*

Heyting-valued structure semantics [A. Dragalin, 1973]

Let's extend a measure of equality to tuples $\boldsymbol{a}, \boldsymbol{b} \in D^k$ in the following way:

$$E(\boldsymbol{a}, \boldsymbol{b}) := E\,(a_1, b_1) \wedge \cdots \wedge E\,(a_k, b_k)$$

and introduce abbreviations:

$$E(\boldsymbol{a}, \boldsymbol{b}) := E\boldsymbol{ab}, E\boldsymbol{aa} := E\boldsymbol{a}.$$

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Proposition

Every measure of equality E has the following properties:

- $E(a, b) \leq E(a, a)$;
- $Eab \leq Eaa$;
- $E\boldsymbol{ab} \wedge E\boldsymbol{bc} \leq E\boldsymbol{ac}$;
- $\bigvee_{a \in D^k} E(\boldsymbol{a}) = 1$.

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A valuation on a H.v.s $F = \langle \Omega, D, E \rangle$ is a map $\varphi : AF_D \to \Omega$ such that for every $P \in PL^n$:

$$\varphi(P(\boldsymbol{a})) \wedge E\,(a_i, b_i) \leq \varphi(P(\boldsymbol{b}))$$

whenever $\boldsymbol{a}, \boldsymbol{b} \in D^k$ and $a =_i b$, which means that $a_k = b_k$ for every $k \neq i$, where a_k and b_k are the k-th elements of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

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We extend the valuation φ to all D-sentences in the following way:

- $\varphi(\bot) := \mathbf{0}$,
- $\varphi(a = b) := E(a, b)$;
- $\varphi(A \vee B) := \varphi(A) \vee \varphi(B)$;
- $\varphi(A \wedge B) := \varphi(A) \wedge \varphi(B)$;
- $\varphi(A \supset B) := \varphi(A) \to \varphi(B)$;
- $\varphi(\exists x A) := \bigvee_{d \in D}(Ed \wedge \varphi(A(d)))$;
- $\varphi(\forall x A) := \bigwedge_{d \in D}(Ed \to \varphi(A(d)))$.

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Definition

A formula A is said to be true in an algebraic model model $\langle F, \varphi \rangle$ if $\varphi(\bar{A}) = 1$, this fact is denoted as $\langle F, \varphi \rangle \models A$. A formula A is said to be valid in an algebraic model F if it is true for every valuation φ on F.

Theorem

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For every Suzuki Algebraic Kripke Sheaf $\mathcal{S} = \langle M, D, P\rangle$ *there exists a Heyting-valued structure* $F_{\mathcal{S}'} = \langle \mathbf{\Omega}_{\mathcal{S}'}, D_{\mathcal{S}'}, E_{\mathcal{S}'}\rangle$ *and for every valuation* V *on* $\mathcal{S}$ *there exists a valuation* $\varphi_{\mathcal{S}'}$ *on* $F_{\mathcal{S}'}$ *which has the same set of valid formulas.*

Algebra of Monotonic Maps

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Note. Every calculation we will need can be done in the algebra $P(0)$.

Consider a Suzuki-defined algebraic Kripke sheaf $\mathcal{S} = \langle \boldsymbol{M}, D, P \rangle$ and an arbitrary valuation V on it.

Let $\boldsymbol{\Omega}_{\mathcal{S}}$ be a set of monotonic maps from M to $P(0)$ with algebraic operations which are defined as follows:

$$\begin{aligned} f \wedge g: &\quad (f \wedge g)(x) := f(x) \cap g(x); \\ f \vee g: &\quad (f \vee g)(x) := f(x) \cup g(x); \\ f \to g: &\quad (f \to g)(x) := \bigcap_{y \succcurlyeq x} (f(y) \to g(y)), \end{aligned}$$

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For every set $\{f_i\}_{i \in I} \subseteq \boldsymbol{\Omega}_{\mathcal{S}}$ greatest lower and least upper bounds are defined termwise as follows:

$$\bigcup \{f_i\}_{i \in I} : \left(\bigcup \{f_i\}_{i \in I} \right)(x) = \bigcup f_i(x)_{i \in I}, \forall x \in \boldsymbol{M}$$

and

$$\bigcap \{f_i\}_{i \in I} : \left(\bigcap \{f_i\}_{i \in I} \right)(x) = \bigcap f_i(x)_{i \in I}, \forall x \in \boldsymbol{M}.$$

Algebra of Monotonic Maps

Partial order: $f \leq^{\mathbf{\Omega}_{\mathcal{S}}} g \Leftrightarrow \forall x \left(f(x) \leq^{P_{\mathbf{0}}} g(x) \right)$, for every $f, g \in \mathbf{\Omega}_{\mathcal{S}}$.

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The least and the greatest elemets of $\mathbf{\Omega}_{\mathcal{S}}$:

$$\mathbf{0} \in \mathbf{\Omega}_{\mathcal{S}} : \mathbf{0}(x) := \mathbf{0}^{P_{\mathbf{0}}}, \forall x \in \boldsymbol{M}$$
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Proposition

The set $\mathbf{\Omega}_{\mathcal{S}}$ with operations $\wedge, \vee, \rightarrow, 0, 1$ and LUB and GLB defined previously is a complete Heyting algebra.

Disjoint Sheaves

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Let $\mathcal{S} = \langle \boldsymbol{M}, D, P \rangle$ be an algebraic Kripke sheaf. Every such Kripke sheaf can be transformed into a *disjoint* Kripke sheaf S' i.e. with disjoint fibers.

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Individuals:	$D'(u) := \{(a,u) \vert a \in D_u\}$
Transitions:	$D'_{uv}((a,u)) := (D_{uv}(a), v)$ for every $v \succcurlyeq u$
Valuation:	$V'(m,(a,u)) := V(m,a)$

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Algebraic Kripke Sheaves $\mathcal{S}$ and $\mathcal{S}'$ are isomorphic.

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Proposition

Algebraic Kripke Sheaves $\mathcal{S}$ and $\mathcal{S}'$ are isomorphic.

Now instead of $\mathcal{S} = \langle \boldsymbol{M}, D, P \rangle$ we will be working with an equivalent disjoint algebraic Kripke sheaf $\mathcal{S}' = \langle \boldsymbol{M}, D', P \rangle$.
And let τ_1 be an isomorphism from $\mathcal{S}$ to $\mathcal{S}'$.

Equivalent H.V.S.

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Consider an algebra $\boldsymbol{\Omega}_{\mathcal{S}'}$ which is an algebra of monotonic maps from the Kripke base $\boldsymbol{M}$ to a complete Heyting algebra $P(0)$. We define the measure of equality on the set $D_{\mathcal{S}'}$ for elements $a, b \in D_{\mathcal{S}'}$, which belong to D'_u and D'_v respectively as follows:

$$E_{\mathcal{S}'}(a, b)(m) := \begin{cases} \mathbf{1}, & \text{if } m \succcurlyeq u, v \text{ and } D'_{um}(a) = D'_{vm}(b) \\ \mathbf{0}, & \text{otherwise} \end{cases} .$$

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The map $E_{\mathcal{S}'}$ is a well-defined measure of existense on $\langle \boldsymbol{\Omega}_{\mathcal{S}'}, D_{\mathcal{S}'} \rangle$.

Equivalent H.V.S.

We now see that $F_{\mathcal{S}'} = \langle \mathbf{\Omega}_{\mathcal{S}'}, D_{\mathcal{S}'}, E_{\mathcal{S}'} \rangle$ is a well-defined Heyting valued structure.

An extension of $E_{\mathcal{S}'}$ for the elements of $D^n_{\mathcal{S}'}$ can be constructed by the same rule that was used in the previous frames in the definition of Heyting-valued structures.

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We now see that $F_{\mathcal{S}'} = \langle \mathbf{\Omega}_{\mathcal{S}'}, D_{\mathcal{S}'}, E_{\mathcal{S}'} \rangle$ is a well-defined Heyting valued structure.

An extension of $E_{\mathcal{S}'}$ for the elements of $D^n_{\mathcal{S}'}$ can be constructed by the same rule that was used in the previous frames in the definition of Heyting-valued structures.

Let structure $F_{\mathcal{S}'} = \langle \mathbf{\Omega}_{\mathcal{S}'}, D_{\mathcal{S}'}, E_{\mathcal{S}'} \rangle$ be an image of $\mathcal{S}' = \langle \boldsymbol{M}, D', P \rangle$ under a map τ_2. Due to the disjoint nature of $\mathcal{S}'$, the map τ_2 is bijective. In order to prove this, we will construct a map ν from $F_{\mathcal{S}'}$ to $\mathcal{S}'$.

Proposition

Algebraic Kripke Sheaves $\mathcal{S}'$ and $\nu(\tau_2(\mathcal{S}')) = \nu(F_{\mathcal{S}'})$ are isomorphic.

Equivalency

We now have the following chain of isomorphic structures:

$$\mathcal{S} = \langle \boldsymbol{M}, D, P \rangle \xrightarrow{\tau_1} \mathcal{S}' = \langle \boldsymbol{M}, D', P \rangle \xrightarrow{\tau_2} F_{\mathcal{S}'} = \langle \boldsymbol{\Omega}_{\mathcal{S}'}, D_{\mathcal{S}'}, E_{\mathcal{S}'} \rangle$$

Whereas $\mathcal{S}$ and $\mathcal{S}'$ have valuations V and V' which yield coinciding sets of valid formulas.

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It remains to define a valuation on Heyting-valued structure $F'_{\mathcal{S}}$.

Generalised Sheaf Valuation

For any formula $A \in IL_{D_{\mathcal{S}}}$, with the set of constants $C(A)$, the field of existence $E(A)$ is defined as:

$$E(A) = \{x \in \boldsymbol{M} \mid \forall c \in C(A) : E_{\mathcal{S}'}(c, c)(x) = 1\}.$$

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For every formula $A \in IL_{D_{\mathcal{S}'}}$ its inheritor A_v can be defined for every point $v \in E(A)$:

$$A_v := \left[D'_{u_{\mathbf{1}}v}\,(a_1) \ldots D'_{u_k v}\,(a_k) \setminus a_1 \ldots a_k\right] A, \text{ where } a_i \in D'_{ui}.$$

Then, an extension V'^{+} of the algebraic Kripke sheaf valuation V' is defined on all formulas as:

$$V'^{+}(m, A) := \begin{cases} V'\,(m, A_m)\,, & \text{if } m \in E(A) \\ \mathbf{0}, & \text{otherwise} \end{cases}.$$

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Map V'^{+} has all the properties of the valuation V' so the latter can be substituted by its more generalised form. The sign $+$ in the notation will now be omitted and V' will actually denote V'^{+}.

H.V.S. Valuation

Definition

Let $\varphi_{\mathcal{S}'} : AF_{D_{\mathcal{S}'}} \to \mathbf{\Omega}_{\mathcal{S}'}$ *be a map defined by:*

$$\varphi_{\mathcal{S}'}(A) = \left\{ \left(m, V'(m, A)\right) \mid m \in M \right\}.$$

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Proposition

A map $\varphi_{\mathcal{S}'}$ *is a valuation on* $F_{\mathcal{S}'} = \langle \mathbf{\Omega}_{\mathcal{S}'}, D_{\mathcal{S}'}, E_{\mathcal{S}'} \rangle$ *in terms of a H.v.s. model i.e. for every two* $\boldsymbol{a}, \boldsymbol{b} \in D^n_{\mathcal{S}'}$ *such that* $\boldsymbol{a} =_i \boldsymbol{b}$ *and an arbitrary predicate* $P \in PL^n$ *the following holds:*

$$\varphi_{\mathcal{S}'}(P(\boldsymbol{a})) \wedge E_{\mathcal{S}'}(a_k, b_k) \leq \varphi_{\mathcal{S}'}(P(\boldsymbol{b})).$$

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Proposition

The extension of the valuation $\varphi_{\mathcal{S}'}$ *to all sentences from* $IL_{D_{\mathcal{S}}}$ *is an H.v.s. valuation.*

Proof of the Theorem

Proposition

For any valuation V' on $\mathcal{S}'$, a valuation $\tau_2(V') := \varphi_{\mathcal{S}'}$ on $F_{\mathcal{S}'}$ has the same set of true formulas.

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Proof.

For any closed formula $A \in IL_{D_{\mathcal{S}'}}$:

$$\begin{aligned} V'\left(0^M, A\right) = 1 &\Leftrightarrow \forall m \in M \;\; V'(m, A) = 1 \\ &\Leftrightarrow \forall m \in \boldsymbol{M} \;\; \varphi_{\mathcal{S}'}(A)(m) = V'(m, A) = 1 \\ &\Leftrightarrow \varphi_{\mathcal{S}'}(A) = 1. \end{aligned}$$

So any formula A is true in $\langle \mathcal{S}', V' \rangle$ iff it is true in $\langle \tau_2\left(\mathcal{S}'\right), \tau_2\left(V'\right) \rangle$. $\square$

Proof of the Theorem

Proposition

Any formula A is valid in $\mathcal{S}'$ iff it is valid in $\tau_2(\mathcal{S}')$.

Proposition

For every Suzuki Algebraic Kripke Sheaf $\mathcal{S} = \langle M, D, P \rangle$ there exists a Heyting-valued structure $F_{\mathcal{S}'} = \langle \mathbf{\Omega}_{\mathcal{S}'}, D_{\mathcal{S}'}, E_{\mathcal{S}'} \rangle$ and for every valuation V on $\mathcal{S}$ there exists a valuation $\varphi_{\mathcal{S}'}$ on $F_{\mathcal{S}'}$ which has the same set of true formulas.

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Proof of the Theorem.

As we know, we have three isomorphic structures

$$\mathcal{S} := \langle \boldsymbol{M}, D, P \rangle \xleftrightarrow{\tau_1} \mathcal{S}' := \left\langle \boldsymbol{M}, D', P \right\rangle \xleftrightarrow{\tau_2} F_{\mathcal{S}'} := \langle \mathbf{\Omega}_{\mathcal{S}'}, D_{\mathcal{S}'}, E_{\mathcal{S}'} \rangle$$

and due to propositions these maps also establish an isomorphism between valuations. □

Fuzzy Sheaves

The generalised disjoint algebraic Kripke sheaf $\mathcal{S} = \langle \boldsymbol{M}, D, P, \asymp \rangle$ where $\asymp$ is a family of maps $\asymp_u \colon D_u^2 \to P(0)$ such that for every $u \in \boldsymbol{M}$:

$$(a \asymp_u b) = (b \asymp_u a)\,, \;\; a, b \in D_u;$$

$$(a \asymp_u a) \neq 0, \;\; a \in D_u;$$

$$(a \asymp_u b) \leq (a \asymp_u a)\,, \;\; a, b \in D_u;$$

$$(a \asymp_u b) \wedge (b \asymp_u b) \leq (a \asymp_u c)\,, \;\; a, b, c \in D_u;$$

$$(D_{uv}(a) \asymp_v D_{uv}(b)) \geq (a \asymp_u b)\,, \;\; a, b \in D_u, v \succcurlyeq u.$$

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$$(a \asymp_u b) \wedge (b \asymp_u b) \leq (a \asymp_u c)\,, \;\; a, b, c \in D_u;$$
$$(D_{uv}(a) \asymp_v D_{uv}(b)) \geq (a \asymp_u b)\,, \;\; a, b \in D_u, v \succcurlyeq u.$$

For every atomic sentence of the type $a = b$ where $a, b \in D_u$ valuation of it is defined as:

$$V(u, a = b) = a \asymp_u b.$$

It is obvious that for every $u, v \in \boldsymbol{M}$ with $v \succcurlyeq u$:

$$V\left(v, D_{uv}(a = b)\right) \geq V(u, a = b).$$

Thank you for attention!