One-Parameter Semigroups of Operators (OPSO) 2021

ON FINAL DYNAMICS OF SEMILINEAR PARABOLIC EQUATIONS

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9 April 2021

Semilinear parabolic equations [1]

First, we consider the abstract dissipative SPE

$$\partial_t u = -Au + F(u) \tag{1}$$

in a *real* infinite-dimensional separable Hilbert space X with the norm $\|\cdot\|$. We assume that the unbounded self-adjoint positive definite linear operator A with domain $\mathcal{D}(A) \subset X$ has a compact resolvent. If $X^\alpha = \mathcal{D}(A^\alpha)$ with $\alpha \geq 0$, then $\|u\|_\alpha = \|A^\alpha u\|$, $X^0 = X$, and $X^1 = \mathcal{D}(A)$.

We let $BC^2(X^\alpha, X)$ denote the class of C^2-smooth mappings $X^\alpha \to X$ that are bounded on balls. We assume that a nonlinear function F belongs to $BC^2(X^\alpha, X)$ for some $\alpha \in [0, 1)$ and equation (1) is *dissipative*, i.e., generates a resolving C^2-semiflow $\{\Phi_t\}_{t\geq 0}$ in the phase space X^α, and

$$\sup \lim_{t\to+\infty} \|\Phi_t u\|_\alpha \leq r \tag{2}$$

for some $r > 0$ uniformly in u from arbitrary bounded subsets $\mathcal{B} \subset X^\alpha$.

1. D. Henry, *Geometric theory of semilinear parabolic equations*, 1981.

Attractors and inertial manifolds [1–3]

The problem of describing the final (at large times) dynamics of SPE by an ordinary differential equation (ODE) in $\mathbb{R}^N$ (**FD reduction**) has been attracting researcher's attraction for a long time. In fact, it is required to separate finitely many "determining" degrees of freedom of an infinite-dimensional dynamical system. In this case, the key geometric object is the so-called (global) *attractor*, i.e., the connected compact invariant set $\mathcal{A} \subset X^\alpha$ that uniformly attracts bounded subsets X^α as $t \to +\infty$. Attractor $\mathcal{A}$ consists of all bounded complete trajectories $\{u(t)\}_{t\in\mathbb{R}}$.

The required ODE can sometimes be implemented as an *inertial form* obtained by restricting the initial equation to an *inertial manifold* (IM), i.e, a finite-dimensional invariant C^1-surface $\mathcal{M} \subset X^\alpha$ containing the attractor and exponentially attracting (with asymptotic phase) all trajectories of SPE as $t \to +\infty$.

1. A.V. Babin and M.I. Vishik, *Attractors of evolution equations*, 1992.

2. R. Temam, *Infinite-dimensional Dynamical Systems in Mechanics and Physics*, 1997.

3. S. Zelik, *Proc. Roy. Soc. Edinburgh, Ser. A*, **144**:6 (2014).

Alternative FD reduction

The theory of inertial manifolds originally encountered systematic difficulties related to the requirement of a sufficient sparseness of the spectrum of the linear operator A in (1). In this connection, several alternative concepts of *finite-dimensional reduction* of SPE have therefore been developed starting from [1–4]. Following [3], we will say that the dynamics of SPE on the attractor (final dynamics) is finite-dimensional (FD) if there exists an ODE in $\mathbb{R}^N$ with Lipschitz vector field, resolving flow $\{\Theta_t\}_{t\in\mathbb{R}}$, and invariant compact set $\mathcal{K} \subset \mathbb{R}^N$ such that the phase semiflows $\{\Phi_t\}_{t\geq 0}$ of equation (1) on $\mathcal{A}$ and $\{\Theta_t\}_{t\geq 0}$ are Lipschitz conjugate on $\mathcal{K}$.

1. A. Eden et al., *Exponential Attractors for Dissipative Evolution Equations*, 1994.

2. J.C. Robinson, *J. Dyn. Differ. Eq.*, **11**:3 (1999).

3. A.V. Romanov, *Sb. Mathematics*, **191**:3 (2000).

4. A.V. Romanov, *Izv. Math.*, **65**:5 (2001).

Alternative FD reduction – 1

The existence of the inertial manifold implies that the dynamics is finite-dimensional on the attractor and, in general, looks like a more attractive property. Indeed, in the first case, the inertial form provides an exponential asymptotics of any solutions of the SPE at large times, and in the second case, we have an ODE reproducing the original dynamics only on the attractor itself. Nevertheless, the fact that the dynamics is finite-dimensional on $\mathcal{A}$ means that the structure of limit regimes of SPE with infinitely many degrees of freedom is no more complicated than the structure of similar regimes of an ODE with Lipschitz vector field in $\mathbb{R}^N$.

There is a hypothesis [1] that the finite-dimensional dynamics on the attractor implies the existence of an inertial manifold. This hypothesis has not yet been confirmed or refuted.

1. S. Zelik, *Proc. Roy. Soc. Edinburgh, Ser. A*, **144**:6 (2014).

1D parabolic equations

In this talk, we consider the problem of whether the final dynamics is finite-dimensional for systems of reaction-diffusion-convection (RDC) equations

$$\partial_t u = D\partial_{xx}u - u + f(x,u)\partial_x u + g(x,u), \tag{3}$$

on the circle $\mathbb{T} = \mathbb{R} \,|\, \mathrm{mod}\, \mathbb{Z}$ with $u = (u_1, ..., u_m)$. The matrix of diffusion coefficients D is assumed to be diagonal, $D = \mathrm{diag}\{d_j\}$, $d_j > 0$. We assume that the matrix function f and the vector function g belong to the class C^∞ on $\mathbb{T} \times \mathbb{R}^m$ and write system (3) in the abstract form (1) with $X = L^2(\mathbb{T}, \mathbb{R}^m)$, self-adjoint positive definite operator $Au = u - Du_{xx}$, and nonlinearity $F : u \to f(x,u)\partial_x u + g(x,u)$. Assume that $\{X^\alpha\}_{\alpha\geq 0}$ is the Hilbert semiscale generated by A and $\mathcal{H}^s = \mathcal{H}^s(\mathbb{T})$ are generalized Sobolev L^2-spaces of scalar functions on $\mathbb{T}$ with arbitrary $s \geq 0$. As the phase space we choose $X^\alpha = \mathcal{H}^{2\alpha}(\mathbb{T}, \mathbb{R}^m)$ with arbitrary $\alpha \in (3/4, 1)$ which is fixed below.

1D parabolic equations - 1

One can establish dissipativity of system (3) in X^α provided that functions f and g are finite in u. Anyway, we further assume that system (3) is dissipative in X^α and there exists the global attractor $\mathcal{A} \subset X^\alpha$, consisting of vector-functions $u = u(x),\ x \in \mathbb{T}$, of class C^1.

The algebraic structure of the "convection matrix" $f = f(x,u),\ f = \{f_{ij}\}, i,j \in \overline{1,m}$, on the convex hull $\operatorname{co}\mathcal{A} \subset X^\alpha$ plays an important role. We will highlight the case of the scalar diffusion matrix $D = dE$, where $d = \text{const}$ and E is the identity matrix.

The final dynamics of systems (3) with scalar diffusion matrix D and spatially homogeneous nonlinearity $f(u)\partial_x u + g(u)$ was studied in [1], and the second restriction seems to be technical. The existence of an inertial manifold was proved in [1] for the scalar equation ($m = 1$), and for $m > 1$, it was proved under the assumption that the function matrix $f(u)$ is diagonal with a unique nonzero element in a convex neighborhood of the attractor. For systems (3) on $[0, 1]$ with Dirichle and Neumann boundary conditions, the existence of an IM was established [2] in the case scalar diffusion matrix D and an arbitrary convection matrix f. The results obtained in [1, 2] are based on a non-local change of the phase variable u which "decreases" the dependence of the nonlinear part (3) on $\partial_x u$ and allows using the well-known in the inertial manifolds theory "spectral gap condition".

1. A. Kostianko and S. Zelik, *Comm. Pure Appl. Anal.*, **17**:1 (2018).
2. A. Kostianko and S. Zelik, *Comm. Pure Appl. Anal.*, **16**:6 (2017).

Main results

Here we study whether the dynamics is finite-dimensional on the attractor, but we do not consider the problem of existence of an inertial manifold for systems of periodic equations (3). At the same time, we consider the case of *nonscalar* diffusion matrix D and prove that the limit dynamics is finite-dimensional for wide classes of systems (3).

Main Theorem [1]. *Assume that RDC system* (3) *is dissipative in* X^α *with* $\alpha \in (3/4, 1)$. *Then the phase dynamics on the attractor* $\mathcal{A} \subset X^\alpha$ *is finite-dimensional if any of the following three conditions is satisfied.*

(A) *The convection matrix* $f = \mathrm{diag}$ *on* $\mathrm{co}\,\mathcal{A}$.

(B) *The diffusion matrix* D *is scalar. For all* $(x, u) \in \mathbb{T} \times \mathrm{co}\,\mathcal{A}$, *the numerical matrices* $f(x, u(x))$ *have* m *distinct real eigenvalues and commute with each other.*

(C) *The diffusion matrix* D *is scalar. For all* $(x, u) \in \mathbb{T} \times \mathrm{co}\,\mathcal{A}$, *the matrices* $f(x, u)$ *are symmetric and commute with each other.*

1. A.V. Romanov, arXiv:2011.01822 (2020).

The discussion of the results

In the case (A), we have $Df = fD$ on $\mathrm{co}\,\mathcal{A}$. The assumptions that the matrices are commutative can conditionally be formulated as **"the consistency of convection with diffusion"** and **"self-consistency of convection"** on the convex hull of the attractor.

In the class of one-dimensional systems (3), was constructed [1] the first example of semilinear parabolic equation of mathematical physics (actually, a system of eight equations with scalar diffusion) that does not demonstrate any finite-dimensional dynamics on the attractor. Surprisingly, the dynamics of 1D RDC systems in the Dirichlet – Neumann cases looks simpler than in the periodic case. **Therefore, the periodic class seems to be a good testing ground for understanding where the finite-dimensional final dynamics of semilinear parabolic equations terminates and the infinite-dimensional final dynamics begins.**

1. A. Kostianko and S. Zelik, *Comm. Pure Appl. Anal.*, **17**:1 (2018).

Possible generalizations

The results of the talk can be generalized to systems on [0,1] of the form

$$\partial_t u = D\partial_{xx}u + f(x, u, \partial_x u) \tag{4}$$

with a smooth vector function $f = (f_1, \ldots, f_m)$. Such systems with various boundary conditions can be reduced (see [1, 2]) to the form (4) by the termwise differentiation and an appropriate change of the variable. The fact that the final dynamics is finite-dimensional for scalar equations (4) was already proved in [3].

The existence of an inertial manifold for systems (4) under Dirichlet and Neumann boundary conditions was proved in [1] in the case $f = f(u, u_x)$ and $D = dE$. One can get final-dimensional dynamics on the attractor for systems (3) under these conditions if $Df = fD$ on $\mathrm{co}\,\mathcal{A}$.

We now turn to substantiation of the Main Theorem results.

1. A. Kostianko and S. Zelik, *Comm. Pure Appl. Anal.*, **16**:6 (2017).
2. A. Kostianko and S. Zelik, *Comm. Pure Appl. Anal.*, **17**:1 (2018).
3. A.V. Romanov, *Izv. Math.*, **65**:5 (2001).

FD reduction method for SPEs

Here are two criteria for the dynamics of SPE (1) to be finite-dimensional on the attractor [1] under the assumption that $F \in BC^2(X^\alpha, X)$.

(Fl) The phase semiflow on $\mathcal{A}$ can be extended to the Lipschitz flow:

$$\|\Phi_t u - \Phi_t v\|_\alpha \le M \|u - v\|_\alpha e^{\kappa|t|}, \quad t \in \mathbb{R},$$

where $M > 0$ and $\kappa \ge 0$ depend only on $\mathcal{A}$.

(GrF) The attractor is a Lipschitz graph over the lowest Fourier modes:

$$\|Pu - Pv\|_\alpha \ge M \|u - v\|_\alpha, \quad u, v \in \mathcal{A}, \quad M = M(\mathcal{A}),$$

for some finite-dimensional spectral projection $P \in \mathcal{L}(X^\alpha)$ of the operator A.

1. A.V. Romanov, *Sb. Mathematics*, **191**:3 (2000).

FD reduction method for SPEs – 1

Let $G(u) = F(u) - Au$ be the vector field of SPE (1). We will use other sufficient conditions for the dynamics to be finite-dimensional on the attractor, which were obtained in [1]. These conditions involve decomposition

$$G(u) - G(v) = (T_0(u,v) - T(u,v))(u - v) \tag{5}$$

on the attractor $\mathcal{A}$, where $T_0(u,v)$ is the field of bounded linear operators in X^α and $T(u,v)$ is the field of unbounded sectorial linear operators on X similar to normal operators $H(u,v) \in \mathcal{L}(X^1, X)$. More accurately,

$$T(u,v) = S^{-1}(u,v)H(u,v)S(u,v), \quad u,v \in \mathcal{A}, \quad S,\, S^{-1} \in \mathcal{L}(X), \tag{6}$$

and $\|T_0(u,v)\| \le K$, $\|S(u,v)\| \le K$, $\|S^{-1}(u,v)\| \le K$ on $\mathcal{A} \times \mathcal{A}$ with $K = K(\mathcal{A})$.

1. A.V. Romanov, *Izv. Math.*, **65**:5 (2001).

FD reduction method for SPEs – 2

We also assume that the *total spectrum*

$$\Sigma_T = \bigcup_{u,v \in \mathcal{A}} \operatorname{spec} T(u,v)$$

is "sufficiently sparse", but this condition is less restrictive than the spectral gap condition in the inertial manifolds theory if $\alpha \neq 0$. **It is known [1], that under some additional technical assumptions on operators T_0 and T in equality (5) the dynamics on $\mathcal{A}$ is finite-dimensional.**

In what follows, we will call the corresponding statement "**The Constructive Theorem**".

1. A.V. Romanov, *Izv. Math.*, **65**:5 (2001).

Vector field decomposition

Our goal is to apply **The Constructive Theorem** to RDC system (3) and to prove that the final dynamics is finite-dimensional. Let $G(u) = -Au + F(u)$ be the vector field in (3). By the integral mean-value theorem for nonlinear operators, we have

$$G(u) - G(v) = -Ah + \left(\int_0^1 \mathrm{D}F(w_\tau)d\tau\right)h \doteq Rh, \quad u, v \in \mathcal{A}, \tag{7}$$

where $h(x) = u(x) - v(x)$, $x \in \mathbb{T}$, and $w_\tau = \tau u + (1-\tau)v$. Here D is the Frechet differentiation. The main idea is related to the change of variable in the linear differential expression Rh for fixed $u, v \in \mathcal{A}$, **which allows one to eliminate the dependence on the derivative $\partial_x h$ in (7).**

Vector field decomposition – 1

To this end, we apply, following the work [1], the transformation $h = U\eta$ to the expression Rh, where the $m \times m$ matrix function $U(x) = U(x; u, v)$, $x \in [0, 1]$, is a solution of the linear Cauchy problem

$$U_x = -\frac{1}{2} D^{-1} B(x) U, \quad U(0) = E, \tag{8}$$

$$B(x) \doteq B(x; u, v) = \int_0^1 f(x, w_\tau(x)) d\tau, \quad u, v \in \mathcal{A}. \tag{9}$$

Here $B = B(x; u, v)$ and $U = U(x; u, v)$ are matrix functions of class C^1 on $[0, 1]$.

Lemma. *If* $Df(x, u) = f(x, u)D$ *for* $x \in \mathbb{T}$, $u \in \operatorname{co}\mathcal{A}$, *then*

$$T(u, v) = U(u, v)(\omega I - D\partial_{xx}) U^{-1}(u, v), \quad u, v \in \mathcal{A}, \tag{10}$$

in decomposition (5), (6).

1. D.A. Kamaev, *Russ. Math. Surv.*, **47**:5 (1992).

The adjoint Cauchy problem

The matrix function $V(x) = U^{-1}(x)$, $V = V(x; u, v)$, $x \in [0, 1]$, $u, v \in \mathcal{A}$, is [1] a solution of the Cauchy problem

$$V_x = \frac{1}{2} V D^{-1} B(x), \quad V(0) = E, \tag{11}$$

adjoint to problem (8). After some calculations we found that the periodic boundary conditions

$$h(1) = h(0), \qquad h_x(1) = h_x(0) \tag{12}$$

become

$$\eta(1) = V(1)\eta(0), \qquad \eta_x(1) = V(1)\eta_x(0), \tag{13}$$

where $V(1) \neq E$ in general. **Change in boundary conditions at transition $h \to \eta$ explains the FD reduction difficulty in our periodic case.**

1. Ju.L. Daleckii and M.G. Krein, *Stability of solutions of differential equations in Banach space*, 1974.

The monodromy operator

Properties of monodromy operators $V(1;u,v) \in \mathcal{L}(\mathbb{R}^m)$ play the key role in our constructions. As a consequence of **The Constructive Theorem** and the Lemma, we have

Proposition. *If RDC system* (3) *is dissipative in* X^α *with* $\alpha \in (3/4, 1)$, *then the phase dynamics on the attractor is finite-dimensional in each of the following two cases.*

(i) *The diffusion matrix* D *is scalar and, for all* $u, v \in \mathcal{A}$, *the monodromy operators* $V(1;u,v)$ *are similar to symmetric positive definite ones with a fixed similarity matrix* $C = C(\mathcal{A})$.

(ii) $Df(x,u) = f(x,u)D$ *on* $\mathbb{T} \times \operatorname{co}\mathcal{A}$ *and the monodromy operators* $V(1;u,v)$ *are similar to diagonal positive definite ones with a fixed similarity matrix* $C = C(\mathcal{A})$.

Applying this Proposition we can prove The Main Theorem (Slide 9).

Some examples

We consider several examples illustrating the above-described theory in terms of properties of the convection matrix f. Here we restrict ourselves to the case of scalar diffusion and assume that RDC system (2) is dissipative in the phase space X^α with $\alpha \in (3/4, 1)$. We assume that all the conditions assumed below on $f = f(x, u)$ are valid for $x \in \mathbb{T}$ and $u = u(x)$, $u \in \operatorname{co} \mathcal{A}$.

Some examples – 1

Example 1. *Assume that* $D = dE$ *and* $f(x,u) = f_1(x,u)Q$ *with a scalar* C^∞*-function* f_1 *and numerical* $m \times m$ *matrix* Q. *Then, the dynamics on the attractor of RDC system* (3) *is finite-dimensional if any of the following two conditions is satisfied:*

(i) *the matrix* Q *has* m *distinct real eigenvalues and* $f_1(x,u(x)) \neq 0$;

(ii) *the matrix* Q *is symmetric.*

Remark. Condition (i) in Example 1 is satisfied, in particular, for upper-triangular and lower-triangular matrices Q with distinct elements on the diagonal. For $m = 2$ and $Q = \{q_{jl}\}$, this condition precisely means that $(q_{11} - q_{22})^2 + 4q_{12}q_{21} > 0$.

Example 2. *The dynamics on the attractor of RDC system* (3) *is finite-dimensional in the case of* $m = 2$, $D = dE$ *and* $f(x,u) = \{f_{ij}(x,u)\}$ *with* $f_{11} = f_{22}$ *and* $f_{12} = f_{21}$.

This is a consequence of **The Main Theorem** and the commutativity of numerical matrices of the form $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$.

Some examples – 2

Example 3. *Assume that* $D = dE$, *the matrix* $f = P_n(Q)$, *where* P_n *is a polynomial of degree* $n \geq 0$ *with functional coefficients* $a_i = a_i(x, u)$, $0 \leq i \leq n$, $a_i \in C^\infty(\mathbb{T} \times \mathbb{R}^m, \mathbb{R})$, *and the numerical matrix* Q *is symmetric. Then the dynamics of the attractor of RDC system* (3) *is finite-dimensional.*

Example 4. *Assume that* $D = dE$ *and* $f = Q(x)$, *where* Q *is a* C^∞ *function matrix. Then the dynamics of RDC system* (3) *is finite-dimensional on the attractor if*

$$Q^{\mathrm{t}}(x) = Q(1 - x), \quad x \in \mathbb{T}, \tag{14}$$

where $(\cdot)^{\mathrm{t}}$ *is the operation of transposition.*

THANKS FOR ATTENTION