# BASIC AUTOMORPHISM GROUPS OF COMPLETE CARTAN FOLIATIONS

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We get sufficient conditions for the full basic automorphism group of a complete Cartan foliation to admit a unique (finite-dimensional) Lie group structure in the category of Cartan foliations. In particular, we obtain sufficient conditions for this group to be discrete. Emphasize that the transverse Cartan geometry may be noneffective. Some estimates of the dimension of this group depending on the transverse geometry are found. Further, we investigate Cartan foliations covered by fibrations and ascertain their specification. Examples of computing the full basic automorphism group of complete Cartan foliations are constructed.

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## 1 Introduction. Main results

The automorphism group is associated with every object of a category. Among central problems there is the question whether the automorphism group can be endowed with a (finite-dimensional) Lie group structure [9].

In the theory of foliations with transverse geometries, morphisms are understood as local diffeomorphisms mapping leaves onto leaves and preserving transverse geometries. The group of all automorphisms of a foliation $(M, F)$ with transverse geometry is denoted by $A(M, F)$. Let $A_L(M, F)$ be the normal subgroup of $A(M, F)$ formed by automorphisms mapping each leaf onto itself. The quotient group $A(M, F)/A_L(M, F)$ is called *the full basic automorphism group* and is denoted by $A_B(M, F)$.

In the investigation of foliations $(M, F)$ with transverse geometry it is natural to put the above problem of the existence of a Lie group structure for the full group $A_B(M, F)$ of basic automorphisms of $(M, F)$.

J. Leslie [11] was the first who solved a similar problem for smooth foliations on compact manifolds. For foliations with complete transversally projectable affine connection this problem was raised by I.V. Belko [2].

Foliations $(M, F)$ with effective transverse rigid geometries were investigated by the first author [17] where an algebraic invariant $\mathfrak{g}_0 = \mathfrak{g}_0(M, F)$, called the structural Lie algebra of $(M, F)$, was constructed and it was proved that $\mathfrak{g}_0 = 0$ is a sufficient condition for the existence of a unique Lie group structure in the full basic automorphism group of this foliation. In the case where $(M, F)$ is a Riemannian foliation, the concept of the structural Lie algebra was introduced previously by P. Molino [12].

Spaces which we call Cartan geometries were introduced by Elie Cartan in the 1920s and were called by him *espaces généralizéd*. The investigation of Cartan geometries (see definition 15) gives the possibility to consider different geometry structures from the unified viewpoint.

We use the notion of Cartan foliation in the sense of R. Blumenthal [3]. We emphasize that the following classes of foliations: parabolic, conformal, Weil, projective, pseudo-Riemannian, Lorentzian, Riemannian foliations and foliations with transverse linear connection belong to Cartan foliations. Therefore, all proved by us theorems and corollaries are valid for all these foliations. Let us denote by $\mathfrak{CF}$ the category of Cartan foliations (the definition is given in subsection 2.1).

In subsection 2.2 we remind the notion of the effective Cartan geometry. It was shown by the first author ([16], Proposition 1) that a Cartan foliation modelled on a noneffective Cartan geometry $\xi = (P(N, H), \omega)$ of type $(G, H)$ admits also an effective transversal Cartan geometry of the type $(G', H')$ where $G' = G/K$, $H' = H/K$ and $K$ is the kernel of the pair $(G, H)$, that is the maximal normal subgroup of $G$ belonging to $H$. Due to this fact we may construct the associated foliated bundle for any Cartan foliation. Note that in ([3], Proposition 3.1) this construction is not correct in general.

By the structural Lie algebra $\mathfrak{g}_0 = \mathfrak{g}_0(M, F)$ of a complete Cartan foliation $(M, F)$ we mean the structural Lie algebra of $(M, F)$ considered with the associated effective transversal Cartan geometry indicated above.

Let us denote by $A(M, F)$ the group of all the automorphisms of the Cartan foliation $(M, F)$ in the category $\mathfrak{CF}$ and by $A_B(M, F)$ the full basic automorphism group.

Recall that a leaf $L$ of a foliation $(M, F)$ is proper if $L$ is an embedded submanifold in $M$. A foliation is called proper [15] if all its leaves are proper. A leaf $L$ is said to be closed if $L$ is a closed subset of $M$. As it is known, any closed leaf is proper.

We get the following theorem about a sufficient condition for the existence a unique Lie group structure in the group of basic automorphisms of complete Cartan foliations and some exact estimates of its dimension.

**Theorem 1.** *Let $(M, F)$ be a complete Cartan foliation modelled on a Cartan geometry of type $\mathfrak{g}/\mathfrak{h}$. If the structural Lie algebra $\mathfrak{g}_0 = \mathfrak{g}_0(M, F)$ is zero, then the basic automorphism group $A_B(M, F)$ of this foliation is a Lie group whose dimension satisfies the inequality*

$$\dim A_B(M, F) \leq \dim(\mathfrak{g}) - \dim(\mathfrak{k}), \tag{1.1}$$

*where $\mathfrak{k}$ is the kernel of the pair $(\mathfrak{g}, \mathfrak{h})$, i.e., the maximal ideal of the Lie algebra $\mathfrak{g}$ belonging to $\mathfrak{h}$, and the Lie group structure in $A_B(M, F)$ is unique.*

*Moreover,*

*(a) if there exists an isolated proper leaf or if the set of proper leaves is countable, then*

$$\dim A_B(M, F) \leq \dim(\mathfrak{h}) - dim(\mathfrak{k}); \tag{1.2}$$

*(b) if the set of proper leaves is countable and dense, then*

$$\dim A_B(M, F) = 0. \tag{1.3}$$

*The estimates* (1.1), (1.2) *are exact and the case of* $(b)$ *is realized.*

In other words, if the associated lifted foliation $(\mathcal{R}, \mathcal{F})$ is formed by fibres of a locally trivial fibration, then the basic automorphism group of $(M, F)$ is a Lie group.

Examples 1 – 3 show the exactness of estimates (1.1) and (1.2). In Example 5 we construct the foliation with the countable dense set of closed leaves and show the realization of the case $(b)$ of Theorem 1.

The following assertion contains sufficient conditions in terms of topology of leaves and their holonomy groups for the basic automorphism group of a Cartan foliation to be a Lie group.

**Corollary 2.** *Let $(M, F)$ be a complete Cartan foliation. If at least one of the following conditions holds:*

*(i) there exists a proper leaf $L$ with discrete holonomy group (in the sense of definition 23);*

*(ii) there is a closed leaf $L$ with discrete holonomy group;*

*(iii) there exists a proper leaf $L$ with finite holonomy group;*

*(iv) there is a closed leaf $L$ with finite holonomy group,*

*then the basic automorphism group $A_B(M, F)$ admits a Lie group structure of dimension at most* $\dim(\mathfrak{h}) - \dim(\mathfrak{k})$, *and this structure is unique.*

In particular, we have

**Corollary 3.** *If $(M, F)$ is a proper complete Cartan foliation, then the basic automorphism group $A_B(M, F)$ admits a unique Lie group structure of dimension at most* $\dim(\mathfrak{g}) - \dim(\mathfrak{k})$.

*Remark* 4. I.V. Belko ([2], Theorem 2) stated that the existence of a closed leaf of a foliation $(M, F)$ with complete transversally projectable affine connection is sufficient for the fact that the basic automorphism group $A_B(M, F)$ to admit a Lie group structure. Example 4 shows that this statement is not true in general.

As it has been indicated above about the existence of the associated effective Cartan geometry, the investigation of the basic automorphism groups of Cartan foliation is reduced to foliations which are modelled on effective Cartan geometries.

**Definition 5.** Let $\kappa : \widetilde{M} \to M$ be the universal covering map. We say that a smooth foliation $(M, F)$ is covered by fibration if the induced foliation $(\widetilde{M}, \widetilde{F})$ is formed by fibres of a locally trivial fibration $\widetilde{r} : \widetilde{M} \to B$.

Further we investigate Cartan foliation covered by fibration.

First we describe the global structure entering the holonomy groups of the Cartan foliations covered by fibrations.

**Theorem 6.** *Let $(M, F)$ be a complete Cartan foliation covered by the fibration $\widetilde{r}$ : $\widetilde{M} \to B$ where $\widetilde{\kappa} : \widetilde{M} \to M$ is the universal covering map. Then*

*(1) there exists a regular covering map $\kappa : \widehat{M} \to M$ such that the induced foliation $\widehat{F}$ is made up of fibres of the locally trivial bundle $r : \widehat{M} \to B$ over a simply connected Cartan manifold $(B, \eta)$;*

*(2) a group $\Psi$ of automorphisms of the Cartan manifold $(B, \eta)$ and epimorphism $\chi : \pi_1(M, x) \to \Psi$ of the fundamental group $\pi_1(M, x)$, $x \in M$, onto $\Psi$ are determined;*

*(3) for all points $y \in M$ and $z \in \kappa^{-1}(y)$ the restriction $\kappa|_{\widehat{L}} : \widehat{L} \to L$ to the leaf $\widehat{L} = \widehat{L}(z)$ of the foliation $(\widehat{M}, \widehat{F})$ is a regular covering map onto the leaf $L = L(y)$, and the group of deck transformations of $\kappa|_{\widehat{L}}$ is isomorphic to the stationary subgroup $\Psi_b$ of the group $\Psi$ at the point $b = r(z) \in B$. Moreover, the subgroup $\Psi_b$ is isomorphic to the holonomy group $\Gamma(L, y)$ of the leaf $L$;*

*(4) the group of deck transformation of $\kappa : \widehat{M} \to M$ is isomorphic to $\Psi$.*

**Definition 7.** The group $\Psi = \Psi(M, F)$ satisfying Theorem 6 is called the *global holonomy group* of the Cartan foliation $(M, F)$ covered by fibration.

We recall the notion of an Ehresmann connection (subsection 3.2). The following two theorems show that the class of Cartan foliations covered by fibrations is large.

**Theorem 8.** *Let $(B, \eta)$ be any simply connected Cartan manifold and $\Psi$ be any subgroup of the automorphism group $Aut(B, \eta)$ of the Cartan manifold $(B, \eta)$. Then there exists a Cartan foliation covered by fibration with the global holonomy group $\Psi$, and statements of Theorem 6 are valid for it.*

**Theorem 9.** *If the transverse Cartan curvature of a complete Cartan foliation $(M, F)$ is equal to zero, then $(M, F)$ is covered by fibration, and statements of Theorem 6 are valid for it.*

*Remark* 10. The first author proved ([18], Theorem 5) that any complete non-Riemannian conformal foliation of codimension $q \geq 3$ is covered by fibration.

The application of Theorem 7 proved by the first author in [17] to Cartan foliations gives us the following interpretation of the structural Lie algebra of Cartan foliations covered by fibrations.

**Theorem 11.** *Let $(M, F)$ be a complete Cartan foliation covered by the fibration $\widetilde{r} : \widetilde{M} \to B$ where $\widetilde{\kappa} : \widetilde{M} \to M$ is the universal covering map. Then the structural Lie algebra $\mathfrak{g}_0 = \mathfrak{g}_0(M, F)$ is isomorphic to the Lie algebra of the Lie group $\overline{\Psi}$, which is the closure of $\Psi$ in the Lie group $Aut(B, \eta)$, where $(B, \eta)$ is the induced Cartan geometry.*

**Corollary 12.** *Under conditions of Theorem 11 the structural Lie algebra $\mathfrak{g}_0(M, F)$ is zero if and only if the global holonomy group $\Psi$ is a discrete subgroup of the Lie group $Aut(B, \eta)$ where $\eta$ is the induced Cartan geometry.*

Our next objective is to find a connection between the basic automorphism group $A_B(M, F)$ of Cartan foliation covered by fibration and its global holonomy group $\Psi$. Application of the foliated bundle over $(M, F)$, Theorems $1, 6$ and $11$ allow us to accomplish this task and to prove the following statement.

**Theorem 13.** *Let $(M, F)$ be a complete Cartan foliation covered by fibration $r : \widehat{M} \to B$ and $(B, \eta)$ is the simply connected Cartan manifold determined in Theorem 6. Suppose that the global holonomy group $\Psi$ is a discrete subgroup in the Lie group $Aut(B, \eta)$. Let $N(\Psi)$ be the normalizer of $\Psi$ in $Aut(B, \eta)$. Then the basic automorphism group $A_B(M, F)$ (in the category of Cartan foliations $\mathfrak{CF}$) is a Lie group which is isomorphic to an open-closed subgroup of the Lie quotient group $N(\Psi)/\Psi$, and* $\dim(A_B(M, F)) = \dim(N(\Psi)/\Psi)$.

In the following theorem we give sufficient conditions for a Cartan foliation to satisfy Theorem 13 and have the basic automorphisms group $A_B(M, F)$ isomorphic to the Lie quotient group $N(\Psi)/\Psi$.

**Theorem 14.** *Let $(M, F)$ be an $\mathfrak{M}$-complete Cartan foliation. If the distribution $\mathfrak{M}$ is integrable, then*

*1. The foliation $(M, F)$ is covered by fibration over the simply connected Cartan manifold $(B, \eta)$, and $(M, F)$ is $(Aut(B, \eta), B)$-foliation.*

*2. If moreover, the normalizer $N(\Psi)$ of global holonomy group $\Psi$ is equal to the centralizer $Z(\Psi)$ of $\Psi$ in the group $Aut(B, \eta)$, then*

$$A_B(M, F) \cong N(\Psi)/\Psi.$$

**Notations** We denote by $\mathfrak{X}(N)$ the Lie algebra of smooth vector fields on a manifold $N$. If $\mathfrak{M}$ is a smooth distribution on $M$, then $\mathfrak{X}_{\mathfrak{M}}(M) := \{X \in \mathfrak{X}(M) \mid X_u \in \mathfrak{M}_u \quad \forall u \in M\}$. If in addition $f : K \to M$ is a submersion, then $f^*\mathfrak{M}$ is the distribution on the manifold $K$ such that $(f^*\mathfrak{M})_z := \{X \in T_zK \mid f_{*z}(X) \in \mathfrak{M}_{f(z)}\}$ where $z \in K$.

Let $\mathfrak{Fol}$ be the category of foliations where morphisms are smooth maps transforming leaves into leaves.

If $\alpha : G_1 \to G_2$ is a group homomorphism, then $Im(\alpha) := \alpha(G_1)$. Let $\cong$ be the denotation of a group isomorphism.

Following to [10] we denote by $P(N,H)$ a principal $H$-bundle over the manifold $N$ with the projection $P \to N$.

# 2 The category of Cartan foliations

## 2.1 The category of Cartan geometries

We recall here the definition of Cartan geometries (see [9],[14] and [6]).

Let $G$ be a Lie group and $H$ is a closed subgroup of $G$. Denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of Lie groups $G$ and $H$ relatively.

**Definition 15.** Let $N$ be a smooth manifold. *A Cartan geometry* on $N$ of type $(G,H)$ is the principal right $H$-bundle $P(N,H)$ with the projection $p : P \to N$ together with a $\mathfrak{g}$-valued 1-form $\omega$ on $P$ satisfying the following conditions:

- $(c_1)$ the map $\omega_w : T_wP \to \mathfrak{g}$ is an isomorphism of vector spaces for every $w \in P$;
- $(c_2)$ $R_h^*\omega = Ad_G(h^{-1})\omega$ for each $h \in H$, where $Ad_G : H \to GL(\mathfrak{g})$ is the joint representation of the Lie subgroup $H$ of $G$ in the Lie algebra $\mathfrak{g}$;
- $(c_3)$ $\omega(A^*) = A$ for every $A \in \mathfrak{h}$, where $A^*$ is the fundamental vector field determined by $A$.

The $\mathfrak{g}$-valued form $\omega$ is called a *Cartan connection form*. This Cartan geometry is denoted by $\xi = (P(N,H),\omega)$. The pair $(N,\xi)$ is called a *Cartan manifold*.

Let $\xi = (P(N,H),\omega)$ and $\xi' = (P'(N',H),\omega')$ be two Cartan geometries with the same structure group $H$. The smooth map $\Gamma : P \to P'$ is called a morphism from $\xi$ to $\xi'$ if $\Gamma^*\omega' = \omega$ and $R_a \circ \Gamma = \Gamma \circ R_a, \ a \in H$. If $\Gamma \in Mor(\xi,\xi')$, then the projection $\gamma : N \to N'$ is defined such that $p' \circ \Gamma = \gamma \circ p$, where $p : P \to N$ and $p' : P' \to N'$ are the projections of the corresponding $H$-bundles. The projection $\gamma$ is called *an automorphism of the Cartan manifold* $(N,\xi)$. Denote by $Aut(N,\xi)$ the full automorphism group of $(N,\xi)$ and by $Aut(\xi)$ the full automorphism group of $\xi$. The category of Cartan geometries is denoted by $\mathfrak{Car}$. Let $A(P,\omega) := \{\Gamma \in Diff(P) \mid \Gamma^*\omega = \omega\}$ be the automorphism group of the parallelizable manifold $(P,\omega)$.

Let $A^H(P,\omega) := \{\Gamma \in A(P,\omega) \mid \Gamma \circ R_a = R_a \circ \Gamma\}$, then $A^H(P,\omega)$ is a closed Lie subgroup of the Lie group $A(P,\omega)$ and $Aut(\xi) = A^H(P,\omega)$ is the automorphism group of Cartan geometry $\xi$. The Lie group epimorphism $\sigma : A^H(P,\omega) \to Aut(N,\xi) : \Gamma \mapsto \gamma$ mapping $\Gamma$ to its projection $\gamma$ is defined.

## 2.2 Effectiveness of Cartan geometries

Remind the notion of *effective Cartan geometry* [14]. Consider a pair Lie groups $(G,H)$, where $H$ is a closed subgroup of $G$. Let $(\mathfrak{g},\mathfrak{h})$ be the appropriate pair of Lie algebras. The maximal ideal $\mathfrak{k}$ of the algebra $\mathfrak{g}$ which is contained in $\mathfrak{h}$ is called *the* *kernel* of pair $(\mathfrak{g}, \mathfrak{h})$. If $\mathfrak{k} = 0$, then the pair $(\mathfrak{g}, \mathfrak{h})$ is called *effective*. Maximal normal subgroup $K$ of the group $G$ belonging to $H$ is called the *kernel* of pair $(G, H)$. As it is known, the Lie algebra of $K$ is equal $\mathfrak{k}$. The Cartan geometry $\xi = (P(M, H), \omega)$ of the type $\mathfrak{g}/\mathfrak{h}$ modelled on pair of the Lie group $(G, H)$, is called *effective* if the kernel $K$ of the pair $(G, H)$ is trivial. As it was proved in ([14], Theorem 4.1), the Cartan geometry $\xi = (P(M, H), \omega)$ of type $\mathfrak{g}/\mathfrak{h}$ is effective if and only if the pair of Lie algebras $(\mathfrak{g}, \mathfrak{h})$ is effective and group $N := \{h \in H \mid Ad_G(h) = id_{\mathfrak{g}}\}$ is trivial.

*Remark* 16. The defined above group epimorphism $\sigma : A^H(P, \omega) \to Aut(N, \xi)$ is isomorphism if and only if the Cartan geometry $\xi$ is effective.

## 2.3 Determination of foliations by $N$-cocycles

Let $M$ be a smooth $n$-dimensional manifold. Let $N$ be a smooth $q$-dimensional manifold the connectivity of which is not assumed. We will call an $(N, \xi)$-*cocycle* on $M$ a family $\{U_i, f_i, \{\gamma_{ij}\}\}_{ij \in J}$ satisfying the following conditions:

1) $\{U_i \mid i \in J\}$ is a covering of the manifold $M$ by open connected subsets $U_i$ of $M$, and $f_i : U_i \to N$ is a submersion with connected fibres;
2) if $U_i \cap U_j \neq \emptyset$, $i, j \in J$, then a isomorphism
$$\gamma_{ij} : f_j(U_i \cap U_j) \to f_i(U_i \cap U_j)$$
is defined, and $\gamma_{ij}$ satisfies the equality $f_i = \gamma_{ij} \circ f_j$ on $U_i \cap U_j$;
3) $\gamma_{ij} \circ \gamma_{jk} = \gamma_{ik}$ if $U_i \cap U_j \cap U_k \neq \emptyset$ for all $x \in U_i \cap U_j \cap U_k$ and $\gamma_{ii} = id_{U_i}$, $i, j, k \in J$.

Two $N$-cocycles are called *equivalent* if there exists an $N$-cocycle containing both of these cocycles. Let $[\{U_i, f_i, \{\gamma_{ij}\}\}_{ij \in J}]$ be the equivalence class of $N$-cocycles on manifold $M$ containing the cocycle $\{U_i, f_i, \{\gamma_{ij}\}\}_{ij \in J}$. Denote by $\Sigma$ the set of fibres (or plaques) of all the submersions $f_i$ of this equivalence class. Note, that $\Sigma$ is the base of some new topology $\tau$ in $M$. The linear connected components of the topological space $(M, \tau)$ form a partition $F := \{L_\alpha \mid \alpha \in \mathfrak{J}\}$ of the manifold $M$ which is called *the foliation of the codimension* $q$, $L_\alpha$ are called its *leaves* and $M$ is the foliated manifold. It is said that foliation $(M, F)$ is determined by an $N$-cocycle $\{U_i, f_i, \{\gamma_{ij}\}\}_{ij \in J}$. Further we denote the foliation by the pair $(M, F)$.

## 2.4 Cartan foliations

Let $N$ be a smooth $q$-dimensional manifold the connectivity of which is not assumed. Let $(M, F)$ be a foliation determined by an $N$-cocycle $\{U_i, f_i, \{\gamma_{ij}\}\}_{ij \in J}$. Let $\xi = (P(N, H), \omega)$ — Cartan geometry of type $\mathfrak{g}/\mathfrak{h}$ with the projection $p : P \to N$. For every open subset $V \subset N$ induced Cartan structure $\xi_V = (P_V(V, H), \omega_V)$ of type $\mathfrak{g}/\mathfrak{h}$ such that $P_V := p^{-1}(V)$ and $\omega_V := \omega|_{P_V}$.

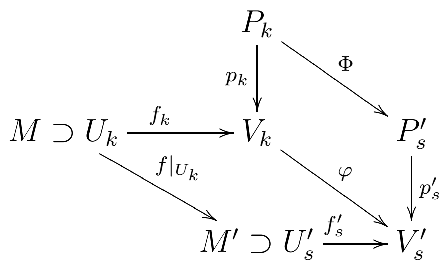
Suppose that for every $\gamma_{ij} : f_j(U_i \cap U_j) \to f_i(U_i \cap U_j)$ there exists an isomorphism $\Gamma_{ij} : \xi_{f_j(U_i \cap U_j)} \to \xi_{f_i(U_i \cap U_j)}$ of the induced Cartan geometries $\xi_{f_j(U_i \cap U_j)}$ and $\xi_{f_i(U_i \cap U_j)}$ with the projection $\gamma_{ij}$. Then the foliation $(M, F)$ is referred as Cartan foliation of type $\mathfrak{g}/\mathfrak{h}$ (or type $(G, H)$) in the since of R. Blumenthal [3]. The Cartan geometry $\xi = (P(N, H), \omega)$ is called *the transverse Cartan geometry* of $(M, F)$. Also it is said that the foliation $(M, F)$ *is modelled on the Cartan manifold* $(N, \xi)$.

*Remark* 17. The first author introduced a different notion of Cartan foliation in [16] that is equivalent to the notion of Cartan foliation in the sense R. Blumenthal if and only if the transverse Cartan geometry is effective.

## 2.5 Morphisms in the category of Cartan foliations

Let $(M,F)$ and $(M',F')$ are Cartan foliations defined by an $(N,\xi)$-cocycle $\eta = \{U_i, f_i, \{\gamma_{ij}\}\}$ and an $(N',\xi')$-cocycle $\eta' = \{U'_r, f'_r, \{\gamma'_{rs}\}\}$ respectively. All objects belonging to $\eta'$ are distinguished by prime. Let $f\colon M \to M'$ be a smooth map which is a local isomorphism in the foliation category $\mathfrak{Fol}$. Hence for any $x \in M$ and $y := f(x)$ there exist neighborhoods $U_k \ni x$ and $U'_s \ni y$ from $\eta$ and $\eta'$ respectively, a diffeomorphism $\varphi\colon V_k \to V'_s$, where $V_k := f_k(U_k)$ and $V'_s := f'_s(U'_s)$, satisfying the relations $f(U_k) = U'_s$ and $\varphi\circ f_k = f'_s \circ f|_{U_k}$. Further we shall use the following notations: $P_k := P|_{V_k},\ \ P'_s := P'|_{V'_s}$ and $p_k := p|_{P_k},\ \ p'_s := p|_{P'_s}$

We say that $f$ preserves *transverse Cartan structure* of $(M,F)$ if every such diffeomorphism $\varphi\colon V_k \to V'_s$ is an isomorphism of the induced Cartan geometries $(V_k, \xi_{V_k})$ and $(V'_s, \xi'_{V'_s})$. This means the existence of isomorphism $\Phi : P_k \to P'_s$ in the category $\mathfrak{Car}$ with the projection $\varphi$, such that the following diagram




is commutative. We emphasize that the indicated above isomorphism $\Phi : P_k \to P'_s$ is not unique if the transverse Cartan geometries are not effective. This notion is well defined, i. e., it does not depend of the choice of neighborhoods $U_k$ and $U'_k$ from the cocycles $\eta$ and $\eta'$.

**Definition 18.** By a *morphism of two Cartan foliations* $(M,F)$ *and* $(M',F')$ we mean a local diffeomorphism $f : M \to M'$ which transforms leaves to leaves and preserves transverse Cartan structure. The category $\mathfrak{CF}$ objects of which are Cartan foliations, morphisms are their morphisms, is called *the category of Cartan foliations.*

# 3 The foliated bundle associated with a Cartan foliation

## 3.1 Associated foliated bundles

The following statement is important for further, and it was proved by the first author ([16], Proposition 1).

**Proposition 19.** *Let $(M,F)$ be a Cartan foliation in the sense of R. Blumenthal with the transverse Cartan geometry $\tilde{\xi} = (\tilde{P}(N,\tilde{H}),\tilde{\omega})$ of type $\tilde{\mathfrak{g}}/\tilde{\mathfrak{h}}$ modeled on a pair of Lie groups $(\tilde{G},\tilde{H})$ with kernel $K$. Then:*

*(i) there exists an effective Cartan geometry $\xi = (P(N,H),\omega)$ of type $\mathfrak{g}/\mathfrak{h}$, modeled on the pair of Lie groups $(G,H)$, where $G = \tilde{G}/K$, $H = \tilde{H}/K$, $\mathfrak{g} = \tilde{\mathfrak{g}}/\mathfrak{k}$, $\mathfrak{h} = \tilde{\mathfrak{h}}/\mathfrak{k}$, and $\mathfrak{k}$ is the kernel of the pair of Lie algebras $(\tilde{\mathfrak{g}},\tilde{\mathfrak{h}})$;*

*(ii) the original foliation $(M,F)$ is a Cartan foliation with an effective transverse Cartan geometry $\xi = (P(N,H),\omega)$.*

Proposition 19 allows us to construct the foliated bundle for an arbitrary Cartan foliation in the sense of R. Blumenthal with noneffective, in general, transverse Cartan geometry $\tilde{\xi}$. Because for effective transverse Cartan geometries the notions of Cartan foliations in the sense of R. Blumenthal and in the sense of [16] are equivalent, we apply Proposition 2 from [16]) to the effective associated transverse Cartan geometry $\xi$ and get Proposition 20. Remind that a Cartan foliation of type $\mathfrak{g}/\mathfrak{o}$ is named transversally parallelizable or $e$-foliation.

**Proposition 20.** *Let $(M, F)$ be a Cartan foliation modelled on Cartan geometry $\tilde{\xi} = (\tilde{P}(N, \tilde{H}), \tilde{\omega})$ of type $\tilde{\mathfrak{g}}/\tilde{\mathfrak{h}}$ and $\xi = (P(\mathcal{N}, H), \omega)$ be the associated effective transverse Cartan geometry of type $(G, H)$, where $G = \tilde{G}/K$, $H = \tilde{H}/K$, $K$ is the kernel of the pair $(\tilde{G}/\tilde{H})$. Then there exists a principal $H$-bundle with a projection $\pi : \mathcal{R} \to M$, $H$-invariant foliation $(\mathcal{R}, \mathcal{F})$ and $\mathfrak{g}$-valued $H$-equivariant 1-form $\beta$ on $\mathcal{R}$ which satisfy the following conditions:*

*(i) $\beta(A^*) = A$ for any $A \in \mathfrak{h}$;*

*(ii) the mapping $\beta_u : T_u\mathcal{R} \to \mathfrak{g}$ $\forall u \in \mathcal{R}$ is surjective, and $ker(\beta_u) = T_u\mathcal{F}$;*

*(iii) the foliation $(\mathcal{R}, \mathcal{F})$ is transversally parallelizable;*

*(iv) the Lie derivative $L_X\beta$ is equal to zero for every vector field $X$ tangent to the foliation $(\mathcal{R}, \mathcal{F})$.*

**Definition 21.** The principal $H$-bundle $\mathcal{R}(M, H)$ satisfying Proposition 20 is said to be *the associated foliated bundle*. The foliation $(\mathcal{R}, \mathcal{F})$ is called *the associated lifted foliation* with the Cartan foliation $(M, F)$.

We denote by $\Gamma(L, x)$, $x \in L$, the germ holonomy group of a leaf $L$ of the foliation usually used in the foliation theory [15]. Next proposition about different interpretations of the holonomy groups of any complete Cartan foliation follows from ([17], Theorem 4).

**Proposition 22.** *Let $(M, F)$ be a complete Cartan foliation, $L = L(x)$ be an arbitrary leaf of this foliation and $\mathcal{L} = \mathcal{L}(u)$, $u \in \pi^{-1}(x)$, be the corresponding leaf of the lifted foliation. Then the germ holonomy group $\Gamma(L, x)$ of leaf $L$ is isomorphic to each of following two groups:*

*(i) the group of deck transformations of the regular covering map $\pi|_{\mathcal{L}} : \mathcal{L} \to L$;*

*(ii) the subgroup $H(\mathcal{L}) = \{a \in H \mid R_a(\mathcal{L}) = \mathcal{L}\}$ of the Lie group $H$.*

*If we change $u$ by an other point $\widetilde{u} \in \pi^{-1}(x)$, then $H(\mathcal{L})$ is changed by the conjugate subgroup $H(\widetilde{\mathcal{L}})$, where $\widetilde{\mathcal{L}} = \widetilde{\mathcal{L}}(\widetilde{u})$, in the group $H$.*

Due to Proposition 22 the following definition is correct.

**Definition 23.** The holonomy group of a complete Cartan foliation $(M, F)$ is called *discrete* if the corresponding group $H(\mathcal{L})$ is a discrete subgroup of the Lie group $H$.

## 3.2 Ehresmann connections for foliations

Let $(M, F)$ be a foliation of codimension $q$ and $\mathfrak{M}$ be a smooth $q$-dimensional distribution on $M$ that is transverse to the foliation $F$. The piecewise smooth integral curves of the distribution $\mathfrak{M}$ are said to be *horizontal,* and the piecewise smooth curves in the leaves are said to be *vertical.* A piecewise smooth mapping $H$ of the square $I_1 \times I_2$ to $M$ is called a *vertical-horizontal homotopy* if the curve $H|_{\{s\}\times I_2}$ is vertical for any $s \in I_1$ and the curve $H|_{I_1\times\{t\}}$ is horizontal for any $t \in I_2$. In this case, the pair of paths $(H|_{I_1\times\{0\}}, H|_{\{0\}\times I_2})$ is called the *base* of $H$. It is well known that there exists at most one vertical-horizontal homotopy with a given base.

A distribution $\mathfrak{M}$ is called an *Ehresmann connection for a foliation* $(M, F)$ (in the sense of R. A. Blumenthal and J. J. Hebda [4]) if, for any pair of paths $(\sigma, h)$ in $M$ with a common initial point $\sigma(0) = h(0)$, where $\sigma$ is a horizontal curve and $h$ is a vertical curve, there exists a vertical-horizontal homotopy $H$ with the base $(\sigma, h)$.

For a simple foliation $F$, i. e., such that it is formed by the fibers of a submersion $r\colon M \to B$, a distribution $\mathfrak{M}$ is an Ehresmann connection for $F$ if and only if $\mathfrak{M}$ is an Ehresmann connection for the submersion $r$, i. e., if and only if any smooth curve in $B$ possesses horizontal lifts.

## 3.3 Completeness of Cartan foliations

Let $(M, F)$ be an arbitrary smooth foliation on a manifold $M$ and $TF$ be the distribution on $M$ formed by the vector spaces tangent to the leaves of the foliation $F$. The vector quotient bundle $TM/TF$ is called the transverse vector bundle of the foliation $(M, F)$. Let us fix an arbitrary smooth distribution $\mathfrak{M}$ on $M$ that is transverse to the foliation $(M, F)$, i. e., $T_xM = T_xF \oplus \mathfrak{M}_x$, $x \in M$, and identify $TM/TF$ with $\mathfrak{M}$.

Let $(M, F)$ be a Cartan foliation and $(\mathcal{R}, \mathcal{F})$ be the lifted foliation with $\mathfrak{g}$-valued 1-form $\beta$ satisfying Proposition 20. It is natural to identify the transverse vector bundle $T\mathcal{R}/T\mathcal{F}$ with the distribution $\widetilde{\mathfrak{M}} := \pi^*\mathfrak{M}$ on $\mathcal{R}$.

**Definition 24.** The Cartan foliation $(M, F)$ is said to be $\mathfrak{M}$-*complete* if any transverse vector field $X \in \mathfrak{X}_{\widetilde{\mathfrak{M}}}(\mathcal{R}, \mathcal{F})$ such that $\beta(X) = \mathrm{const}$ is complete. A Cartan foliation $(M, F)$ of arbitrary codimension $q$ is said to be *complete* if there exists a smooth $q$-dimensional transverse distribution $\mathfrak{M}$ on $M$ such that $(M, F)$ is $\mathfrak{M}$-complete [16].

In other words, $(M, F)$ is an $\mathfrak{M}$-complete foliation if and only if the lifted $e$-foliation $(\mathcal{R}, \mathcal{F})$ is complete with respect to the distribution $\widetilde{\mathfrak{M}} = \pi^*\mathfrak{M}$ in the sense of L. Conlon [7].

The following statement was proved by the first author ([16], Proposition 3).

**Proposition 25.** *If $(M, F)$ is an $\mathfrak{M}$-complete Cartan foliation, then $\mathfrak{M}$ is an Ehresmann connection for this foliation.*

## 3.4 Structural algebras Lie of Lie foliations with dense leaves

Let $(M, F)$ be a Lie foliation with dense leaves. It is the Cartan foliation of a type $\mathfrak{g}_0/\mathfrak{o}$. J. Leslie [11] was the first who observed that the Lie algebra $\mathfrak{g}_0$ of that foliation is invariant in the category of foliations $\mathfrak{Fol}$.

**Definition 26.** The Lie algebra $\mathfrak{g}_0$ of the Lie foliation $(M, F)$ with dense leaves is called the *structural Lie algebra* of $(M, F)$.

### 3.5 Structural Lie algebras of Cartan foliations

Applying of the relevant results of P. Molino [12] and of L. Conlon [7] on complete $e$-foliations we obtain the following theorem.

**Theorem 27.** *Let $(M, F)$ be a complete Cartan foliation and $(\mathcal{R}, \mathcal{F})$ be the associated lifted e-foliation. Then:*

*(i) the closure of the leaves of the foliation $\mathcal{F}$ are fibers of a certain locally trivial fibration $\pi_b \colon \mathcal{R} \to W$;*

*(ii) the foliation $(\overline{\mathcal{L}}, \mathcal{F}|_{\overline{\mathcal{L}}})$ induced on the closure $\overline{\mathcal{L}}$ is a Lie foliation with dense leaves with the structural Lie algebra $\mathfrak{g}_0$, that is the same for any $\mathcal{L} \in \mathcal{F}$.*

**Definition 28.** The structural Lie algebra $\mathfrak{g}_0$ of the Lie foliation $(\overline{\mathcal{L}}, \mathcal{F}|_{\overline{\mathcal{L}}})$ is called *the structural Lie algebra* of the complete foliation $(M, F)$ and is denoted by $\mathfrak{g}_0 = \mathfrak{g}_0(M, F)$.

If $(M, F)$ is a Riemannian foliation on a compact manifold, this notion coincides with the notion of the structural Lie algebra in the sense of P. Molino [12].

**Definition 29.** The fibration $\pi_b \colon \mathcal{R} \to W$ satisfying Theorem 27 is called the *basic fibration* for $(M, F)$.

## 4 Basic automorphisms of Cartan foliations

### 4.1 Groups of basic automorphisms of Cartan foliations

**Definition 30.** Let $A(M, F)$ be the full automorphism group of a Cartan foliation $(M, F)$ in the category of Cartan foliation $\mathfrak{CF}$. The group

$$A_L(M, F) := \{f \in A(M, F) \mid f(L_\alpha) = L_\alpha \ \ \forall L_\alpha \in F\}$$

is a normal subgroup of $A(M, F)$ which is called the *leaf automorphism group* of $(M, F)$. The quotient group $A(M, F)/A_L(M, F)$ is called *the basic automorphism group* and is denoted by $A_B(M, F)$.

Let us emphasize, that the basic automorphism group $A_B(M, F)$ of a Cartan foliation $(M, F)$ is an invariant of this foliation in the category $\mathfrak{CF}$.

### 4.2 Properties of the basic automorphism groups of Cartan foliations

For a Cartan foliation with effective transverse Cartan geometry Proposition 31 follows from ([17], Proposition 9).

**Proposition 31.** *Let $(M, F)$ be a Cartan foliation modelled on an effective Cartan geometry. Let $A^H(\mathcal{R}, \mathcal{F}) := \{h \in A(\mathcal{R}, \mathcal{F}) \,|\, R_a \circ h = h \circ R_a \ \forall a \in H\}$, $A^H_L(\mathcal{R}, \mathcal{F}) := \{h \in A_L(\mathcal{R}, \mathcal{F}) \,|\, R_a \circ h = h \circ R_a \ \forall a \in H\}$ and $A^H_B(\mathcal{R}, \mathcal{F})$ be the quotient group $A^H(\mathcal{R}, \mathcal{F})/A^H_L(\mathcal{R}, \mathcal{F})$.*

*Then there exists the group isomorphism $\delta : A^H_B(\mathcal{R}, \mathcal{F}) \to A_B(M, F)$ satisfying the commutative diagram*

$$\begin{array}{ccc} A^H(\mathcal{R}, \mathcal{F}) & \xrightarrow{\ \mu\ } & A(M, F) \\ {\scriptstyle \alpha^H}\downarrow & & \downarrow{\scriptstyle \alpha} \\ A^H_B(\mathcal{R}, \mathcal{F}) & \xrightarrow{\ \delta\ } & A_B(M, F). \end{array}$$

*where $\alpha^H$ and $\alpha$ are the group epimorphisms onto the indicated quotient groups.*

Assume that the structural Lie algebra $\mathfrak{g}_0 = \mathfrak{g}_0(M, F)$ is zero for a complete Cartan foliation $(M, F)$. Then the lifted foliation $(\mathcal{R}, \mathcal{F})$ is formed by fibres of the locally trivial fibration $\pi_b : \mathcal{R} \to W$ and the $\mathfrak{g}$-valued 1-form $\beta$ on $\mathcal{R}$ is determined according to Proposition 20. In compliance with ([16], Proposition 4) the map

$$W \times H \to W : (w, a) \mapsto \pi_b(R_a(u)) \ \ \forall\, (w, a) \in W \times H, u \in \pi_b^{-1}(w),$$

defines a locally free action of the Lie group $H$ on the basic manifold $W$, and the orbits space $W/H$ is homeomorphic to the leaf space $M/F$. Identify $W/H$ with $M/F$. Connected components of the orbits of this action form a regular foliation $(W, F^H)$. The equality $\pi_b^*\widetilde{\beta} := \beta$ defines an $\mathfrak{g}$-valued 1-form $\widetilde{\beta}$ on $W$ such that $\widetilde{\beta}(A_W^*) = A$, where $A_W^*$ is the fundamental vector field on $W$ defined by $A \in \mathfrak{h} \subset \mathfrak{g}$.

Denote by $A(W, \widetilde{\beta})$ the Lie group of automorphisms of the parallelizable manifold $(W, \widetilde{\beta})$, i.e., $A(W, \widetilde{\beta}) = \{f \in Diff(W) \,|\, f^*\widetilde{\beta} = \widetilde{\beta}\}$. Let

$$A^H(W, \widetilde{\beta}) = \{f \in A(W, \widetilde{\beta}) \,|\, f \circ R_a = R_a \circ f, a \in H\}.$$

Then $A^H(W, \widetilde{\beta})$ and its unity component $A_e^H(W, \widetilde{\beta})$ are Lie groups as closed subgroups of $A(W, \widetilde{\beta})$.

**Proposition 32.** *Let $(M, F)$ be a complete Cartan foliation with an effective transverse geometry and $\mathfrak{g}_0 = \mathfrak{g}_0(M, F) = 0$, $(W, \widetilde{\beta})$ be the corresponding parallelisable basic manifold for the lifted foliation $(\mathcal{R}, \mathcal{F})$, where $W = \mathcal{R}/\mathcal{F}$. Then there exists a Lie group monomorphism*

$$\nu : A_B^H(\mathcal{R}, \mathcal{F}) \to A^H(W, \widetilde{\beta}) : h \cdot A_L^H(\mathcal{R}, \mathcal{F}) \mapsto \widetilde{h},$$

*where $h \in A^H(\mathcal{R}, \mathcal{F})$ and $\widetilde{h}$ is the projection of $h$ with respect to the basic fibration $\pi_b : \mathcal{R} \to W$, and $Im(\nu)$ is an open-closed Lie subgroup of $A^H(W, \widetilde{\beta})$.*

*Consequently, $\varepsilon = \nu \circ \delta^{-1} : A_B(M, F) \to A^H(W, \widetilde{\beta})$ is a Lie group monomorphism, and $Im(\varepsilon)$ is an open-closed Lie subgroup of $A^H(W, \widetilde{\beta})$.*

*Доказательство.* By condition $\mathfrak{g}_0(M, F) = 0$ and the lifted foliation $(\mathcal{R}, \mathcal{F})$ is formed by fibres of the submersion $\pi_b : \mathcal{R} \to W$. Then every $h \in A^H(\mathcal{R}, \mathcal{F})$ induces $\widetilde{h} \in A^H(W, \widetilde{\beta})$, and the map $\rho : A^H(\mathcal{R}, \mathcal{F}) \to A^H(W, \widetilde{\beta})$ is defined. It is clear that $\rho$ is a group homomorphism with the kernel $Ker(\rho) = A_L^H(\mathcal{R}, \mathcal{F})$. As $A_L^H(\mathcal{R}, \mathcal{F})$ is the normal subgroup of $A^H(\mathcal{R}, \mathcal{F})$, there exists a group monomorphism $\nu : A_B^H(\mathcal{R}, \mathcal{F}) \to A^H(W, \widetilde{\beta})$ satisfying the equality $\rho := \nu \circ \alpha^H$, where $\alpha^H : A^H(\mathcal{R}, \mathcal{F}) \to A_B^H(\mathcal{R}, \mathcal{F})$ is the natural projection onto the quotient group $A_B^H(\mathcal{R}, \mathcal{F}) = A^H(\mathcal{R}, \mathcal{F})/A_L^H(\mathcal{R}, \mathcal{F})$.

Suppose that $A^H(W, \widetilde{\beta})$ is a discrete Lie group, then $A_B^H(\mathcal{R}, \mathcal{F})$ is also discrete Lie group and the required statement is true.

Further we assume that $\dim(A^H(W)) \geq 1$.

Let $\mathfrak{a}$ be the Lie algebra of the Lie group $A^H(W, \widetilde{\beta})$. Let $B^*$ be the fundamental vector field defined by $B \in \mathfrak{a}$. Hence $X := B^*$ is a complete vector field on $W$, which defines an 1-parameter group $\phi_t^X$, $t \in (-\infty, +\infty)$, of transformations from $A^H(W, \widetilde{\beta})$.

Let $f$ be any element from the identity component $A_e^H(W, \widetilde{\beta})$ of the Lie group $A^H(W, \widetilde{\beta})$. Then there exists $B \in \mathfrak{a}$ and $t_0 \in (-\infty, +\infty)$ such that $f = \phi_{t_0}^X$ where $X = B^*$. Since $\pi_b : \mathcal{R} \to W$ is the submersion with the Ehresmann connection $\widetilde{\mathfrak{M}}$, where $\widetilde{\mathfrak{M}} = \pi^*\mathfrak{M}$, there exists the unique vector field $Y \in \mathfrak{X}_{\widetilde{\mathfrak{M}}}(\mathcal{R})$ such that $\pi_{b*}Y = X$. The completeness of the vector field $X$ implies the completeness of the vector field $Y$. Hence $Y$ defines a 1-parameter group $\psi_t^Y$, $t \in (-\infty, +\infty)$, of diffeomorphisms of the manifold $\mathcal{R}$.

Let us shows that $\psi_t^Y \in A_e^H(\mathcal{R},\mathcal{F})$ for all $t\in(-\infty,+\infty)$, i.e., we have to check the validity of the following facts: 1) the map $\psi_t^Y$, $t\in(-\infty,+\infty)$, is an isomorphism of $(\mathcal{R},\mathcal{F})$ in the category $\mathfrak{Fol}$; 2) $L_Y\beta = 0$; 3) $L_Y A^* = 0$ for all $A\in\mathfrak{h}$.

1) The equality $\pi_{b*}Y = X$ implies the relation $\pi_b\circ\psi_t^Y = \psi_t^X\circ\pi_b$ for any fixed $t \in (-\infty,\ \infty)$, hence $\psi_t^Y(\pi_b^{-1}(v)) = \pi_b^{-1}(\psi_t^X(v))$ for all $v\in W$, and $\psi_t^Y$ is the isomorphism the lifted foliation $(\mathcal{R},\mathcal{F})$ in the category $\mathfrak{Fol}$.

2) Take arbitrary $u\in\mathcal{R}$ and $Z_0\in\widetilde{\mathfrak{M}}_u$. There is the unique vector field $Z\in\mathfrak{X}_{\widetilde{\mathfrak{M}}}(\mathcal{R})$ such that $Z|_u = Z_0$ and $\beta(Z) = \beta(Z_0) = const$. Put $Z_W := \pi_{b^*}Z$ and apply the following formula [9]

$$(L_X\widetilde{\beta})(Z_W) = X(\widetilde{\beta}(Z_W)) - \widetilde{\beta}([X,Z_W]). \tag{4.1}$$

The relation $\beta = \widetilde{\beta}\circ\pi_{b^*}$ implies that $\widetilde{\beta}(Z_W) = \beta(Z_0) = const$, so $X(\widetilde{\beta}(Z_W)) = 0$. By the choice of $X$, $\phi_t^X \in A^H(W,\widetilde{\beta})$, therefore we have $L_X\widetilde{\beta} = 0$. Hence the equality (4.1) gives

$$\widetilde{\beta}([X,Z_W]) = 0. \tag{4.2}$$

In the formula

$$(L_Y\beta)(Z) = Y(\beta(Z)) - \beta([Y,Z]). \tag{4.3}$$

the first term $Y(\beta(Z)) = 0$, because $\beta(Z) = const$. The relations $\beta = \widetilde{\beta}\circ\pi_{b^*}$ and (4.2) imply the following of equalities:

$$\beta([Y,Z]) = \widetilde{\beta}(\pi_{b^*}[Y,Z]) = \widetilde{\beta}([\pi_{b^*}Y,\ \pi_{b^*}Z]) = \widetilde{\beta}([X,Z_W]) = 0.$$

Therefore (4.3) implies that $(L_Y\beta)(Z) = 0$. Thus, $L_Y\beta = 0$.

3) Denote by $(W,F^H)$ the foliation formed by the connected components of orbits of the action $\Phi^W$ the defined above of $H$ on $W$. Let $(\mathcal{R},\mathcal{F}^H)$ be the foliation formed by the connected components of orbits of the Lie group $H$ on $\mathcal{R}$.

At any point $u\in\mathcal{R}$ there is an neighbourhood $\mathcal{W}$ foliated with respect to both foliations $(\mathcal{R},\mathcal{F})$ and $(\mathcal{R},\mathcal{F}^H)$ which meets each leaf of these foliation in at most one connected subset. We can suppose that the basic fibration $\pi_b:\mathcal{R}\to W$ is the trivial in the neighbourhood $\pi_b^{-1}(\mathcal{V})$, where $\mathcal{V} := \pi_b(\mathcal{W})$. Put $U = \pi(\mathcal{W})$. Let $r: U\to U/(\mathcal{F}|_U)$ and $s:\mathcal{V}\to\mathcal{V}/(\mathcal{F}^H|_{\mathcal{V}})$ be the quotient maps. We can identify $U/(\mathcal{F}|_U)$ and $\mathcal{V}/(\mathcal{F}^H|_{\mathcal{V}})$ with the manifold $V$ such that the diagram

$$\begin{array}{ccc} \mathcal{W} & \xrightarrow{\pi_b} & \mathcal{V} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle s} \\ U & \xrightarrow{r} & V, \end{array} \tag{4.4}$$

where the restriction of $\pi$ and $\pi_b$ onto $\mathcal{W}$ are denote by the same letters, is commutative. Without lost the generality, we can assume that $\mathfrak{M}|_U$ is an Ehresmann connection for the submersion $r$ and $\widetilde{\mathfrak{M}}|_{\mathcal{W}}$ is an Ehresmann connection for the submersion $\pi_b$.

Let $A_W^*$ be the fundamental vector field on $W$ defined by $A\in\mathfrak{h}$. Since the action $\Phi^H$ of the Lie group $H$ on $W$ is defined any element $A\in\mathfrak{h}$ defines 1-parametric group tangent vector field to which is called the fundamental vector field and is denoted by $A_W^*$. By the choice of $X := B^*$ on $W$ for any $A\in\mathfrak{h}$ we have the equality $L_X A_W^* = 0$, i.e., $[A_W^*, X] = 0$. Since the fundamental vector fields $A_W^*$ span the tangent spaces to the leaves of the foliation $(W,F^H)$, it is not difficult to cheek that $X$ is the foliated vector field for this foliation. Hence the vector field $X_V := s_*X|_{\mathcal{V}}$ is well defined. There is the unique vector field $Y_U\in\mathfrak{X}_{\mathfrak{M}}(U)$ such that $r_*Y_U = X_V$. In other words, $Y_U$ is the $\mathfrak{M}$-horizontal lift of $X_V$. The commutative diagram (4.4) implies the relation

$\pi_* Y_{\mathcal{W}} = Y_U$, hence $Y$ is a foliated vector field with respect to the foliation $(\mathcal{R}, \mathcal{F}^H)$. Therefore,

$$[A^*, Y] \in \mathfrak{X}_{\mathcal{F}^H}(\mathcal{R}). \tag{4.5}$$

Due to the equalities $\beta(A^*) = \widetilde{\beta}(A^*_W) = A$, the vector field $A^*$ is foliated with respect to $(\mathcal{R}, \mathcal{F})$. So we have the following chain of equalities

$$\pi_{b^*}[A^*, Y] = [\pi_{b^*} A^*, \pi_{b^*} Y] = [A^*_W, X] = 0,$$

hence,

$$[A^*, Y] \in \mathfrak{X}_{\mathcal{F}}(\mathcal{R}). \tag{4.6}$$

The relations (4.5) and (4.6) imply the equality $[A^*, Y] = 0$ for all $A \in \mathfrak{h}$. This ends the check that $\phi_t^Y \in A_e^H(\mathcal{R}, \mathcal{F})$, and consequently $\phi_t^X \in A_e^H(W, \widetilde{\beta})$. Thus, we proved the inclusion $A_e^H(W, \widetilde{\beta}) \subset Im(\rho) = Im(\nu)$. Therefore $Im(\nu)$ is an open-closed Lie subgroup of $A^H(W, \widetilde{\beta})$

Therefore, $\varepsilon = \nu \circ \delta^{-1} : A_B^H(\mathcal{R}, \mathcal{F}) \to A^H(W, \widetilde{\beta}) : \widehat{f} \cdot A_L^H(\mathcal{R}, \mathcal{F}) \mapsto f$ is the monomorphism of Lie groups, and $Im(\varepsilon) = \varepsilon(A_B^H(\mathcal{R}, \mathcal{F}))$ is an open-closed Lie subgroup of the Lie group of $A^H(W, \widetilde{\beta})$. □

### 4.3 Proof Theorem 1

Let $(M, F)$ be a Cartan foliation modelled on a Cartan geometry $\widetilde{\xi} = (\widetilde{P}(N, H), \widetilde{\omega})$ of type $(\widetilde{G}, \widetilde{H})$ and the Lie group $K$ be the kernel of the pair Lie groups $(\widetilde{G}, \widetilde{H})$, $\mathfrak{k}$ be the Lie algebra of $K$. Then the associated effective Cartan geometry $\xi = (P(M, H), \omega)$ of type $(G, H)$, where $G = \widetilde{G}/K$, $H = \widetilde{H}/K$, is defined. Here $\omega$ is the $\mathfrak{g}$-valued 1-form on $P$, where $\mathfrak{g} = \widetilde{\mathfrak{g}}/\mathfrak{k}$. According to Proposition 20 the associated foliated bundle $\mathcal{R}(M, H)$ with the lifted foliation $(\mathcal{R}, \mathcal{F})$ and the projection $\pi : \mathcal{R} \to M$ are defined.

If $\dim(A^H(W, \widetilde{\beta})) = 0$, then according to Proposition 32, $A_B(M, F)$ is a discrete Lie group, hence the estimates (1.1) and (1.2) are valid.

Suppose now that $\dim(A_B(M, F)) \geq 1$.

Denote by $A_B(M, F)_e$ the unite component of the Lie group $A_B(M, F)$. According to Proposition 32, $\varepsilon|_{A_B(M,F)_e} : A_B(M, F)_e \to A_e^H(W, \widetilde{\beta})$ is the group isomorphism. Therefore the basic automorphism group $A_B(M, F)$ admits a Lie group structure of the dimension not more then $\dim(W) = \dim(\mathfrak{g})$. Because $\mathfrak{g} = \widetilde{\mathfrak{g}}/\mathfrak{k}$ we have $\dim(\mathfrak{g}) = \dim(\widetilde{\mathfrak{g}}) - \dim(\mathfrak{k})$. Hence the dimension of the Lie group $A_B(M, F)$ satisfies the following inequality $\dim(A_B(M, F)) \leq \dim(\widetilde{\mathfrak{g}}) - \dim(\mathfrak{k})$. As the Lie group $A_B(M, F)_e$ is realized as a closed subgroup of the automorphism Lie group $A(W, \widetilde{\beta})$ of a parallelizable manifold, then it admits a unique topology and a unique smooth structure that make it into a Lie group ([1], Proposition 1). The same is valid for the group $A_B(M, F)$.

$(a)$ Let $s : W \to W/H$ be the projection onto the orbit space. Assume now that there exists an isolated proper leaf $L$ of the foliation $(M, F)$. Let $x \in L$, $u = \pi^{-1}(x)$ and $w = \pi_b(u) \in W$. Observe, that any automorphism of a foliation transforms a proper leaf to the corresponding proper leaf. Then $q(L) = s(w)$ and the orbit $A_e^H(W, \widetilde{\beta}) \cdot w$ belongs to $s^{-1}(s(w))$. Consequently we have $\dim(A_B(M, F)) = \dim(A_e^H(W, \widetilde{\beta}) \cdot w) \leq \dim(H) = \dim(\mathfrak{h}) = \dim(\widetilde{\mathfrak{h}}) - \dim(\mathfrak{k})$.

Thus $\dim(A_B(M, F)) \leq \dim(\widetilde{\mathfrak{h}}) - \dim(\mathfrak{k})$.

Suppose now that the set of proper leaves of $(M, F)$ is countable (nonempty). Consider any 1-parametric group $\varphi_t$, $t \in (-\infty, +\infty)$ from the Lie group $A^H(W, \widetilde{\beta}) \cong A_B(M, F)_e$. Let $L = L(x)$ be any leaf, $u = \pi^{-1}(x)$ and $w = \pi_b(u) \in W$. Let $w \cdot H$ be the orbit of $w$ relatively $H$. Since for any fixed $t$ the automorphism $\varphi_t$ transforms a proper leaf $L$ to the proper leaf $\varphi_t(L)$ the countability of the set proper leaves implies that $\varphi_t(w \cdot H) = w \cdot H$. Hence, by analogy with the previous case we have the estimate (1.2).

(*b*) Now we suppose that the set of proper leaves $\{L_n \,|\, n \in \mathbb{N}\}$ be countable and dense. Let $x_n \in L_n$, $u_n = \pi^{-1}(x_n)$ and $w_n = \pi_b(u_n) \in W$. Let us assume that $\dim(A_B(M,F)) \geq 1$. Let $\varphi_t$, $t \in (-\infty, +\infty)$, be any 1-parametric subgroup of the automorphism group $A^H(W, \widetilde{\beta}) \cong A_B(M,F)$. As it was proved above, it is necessary $\varphi_t(w_n \cdot H) = w_n \cdot H$ for all $t \in (-\infty, +\infty)$ and $n \in \mathbb{N}$. Remark that the leaf space $M/F$ homeomorphic the orbit space $W/H$. Denote by $\widetilde{\varphi}_t$ the induced 1-parametric group of homeomorphisms of the leaf space $M/F$. Therefore, for each $t \in (-\infty, +\infty)$, $t \neq 0$, the homeomorphism $\widetilde{\varphi}_t$ has dense subset $\{[L_n] \,|\, [L_n] = s(w_n \cdot H),\, n \in \mathbb{N}\}$ of fixed points in the leaf space $M/F = W/H$.

Due to continuity and openness of the projection $q : M \to M/F$, the leaf space $M/F$ is a first-countable space, that is every its point has a countable neighbourhood basis. Then $\widetilde{\varphi}_t$ is sequentially continuous. Therefore the existence of dense subset of fixed points of the homeomorphism $\widetilde{\varphi}_t$ implies $\widetilde{\varphi}_t = id_{M/F}$. Hence $\varphi_t = id_W$ that contradicts to the assumption.

Thus, $\dim(A_B(M,F)) = 0$ and (1.3) is proved.

### 4.4 Proof of Corollary 2

Observe that the existence of a proper leaf $L$ with discrete holonomy group guarantees the equality $\mathfrak{g}_0(M,F) = 0$.

Remark that any closed leaf of a foliation is proper and each finite holonomy group is a discrete one. Hence we have implications $(iv) \Rightarrow (iii) \Rightarrow (ii) \Rightarrow (i)$. Thus, applying Theorem 1 we get the required assertion.

### 4.5 Proof of Corollary 3

It is well known that any foliation has leaves without holonomy. Therefore, the Corollary 3 follows from the item (*iii*) of Corollary 2.

## 5 The structure of Cartan foliations covered by fibrations

### 5.1 $(G,B)$-foliations

Let $B$ be a connected smooth manifold and $G$ be a Lie group of diffeomorphisms of $B$. The group $G$ is said *to act quasi-analytically on* $B$ if, for any open subset $V$ in $B$ and an element $g \in G$ the equality $g|_V = id_V$ implies $g = e$, where $e$ is the identity transformation of $B$.

**Definition 33.** Assume that the Lie group $G$ of diffeomorphisms of a manifold $B$ acts on $N$ quasi-analytically. A foliation $(M,F)$ defined by an $B$-cocycle $\{U_i, f_i, \{\gamma_{ij}\}\}_{i,j \in J}$ is called a $(G,B)$-foliation if for any $U_i \cap U_j \neq \emptyset$, $i,j \in J$, there is an element $g \in G$ such that $\gamma_{ij} = g|_{f_j(U_i \cap U_j)}$.

### 5.2 Proof of Theorem 6

The following lemma will be useful for us.

**Lemma 34.** *Let $\eta = (P(B,H),\beta)$ be an effective Cartan geometry on a connected manifold $B$ and $\Phi$ be a group of automorphisms of $(B,\eta)$. Then the group $\Phi$ acts quasi-analytically on $B$.*

*Доказательство.* Suppose that there are $\gamma \in \Phi$ and an open subset $U \subset B$ such that $\gamma|_U = id_U$. Then there exists a unique $\Gamma \in Aut(\xi)$ lying over $\gamma$. Let $p : P \to B$ be the projection of the $H$-bundle $P(B,H)$. Observe that any connected component of $P$ is a connected component of some point $v \in p^{-1}(U)$, i. e. it may be represented in the form $P_v$. The effectiveness of the Cartan geometry $\eta$ implies $\Gamma|_{p^{-1}(U)} = id_{p^{-1}(U)}$. Hence $\Gamma$ preserves each connected component $P_v$ of $P$. Because $\Gamma$ is an isomorphism of the connected parallelizable manifold $(P_v, \beta|_{P_v})$ and $\Gamma(v) = v$, then it is necessary $\Gamma|_{P_v} = id_{P_v}$. Therefore $\Gamma = id_P$ and $\gamma = id_B$.

Thus, the group $\Phi$ acts quasi-analytically on $B$. □

Suppose that a Cartan foliation $(M,F)$ modelled on the effective Cartan geometry $\xi = (P(N,H),\omega)$ is covered by a fibration $\widetilde{r} : \widetilde{M} \to B$, where $\widetilde{\kappa} : \widetilde{M} \to M$ is the universal covering map. The fibration $\widetilde{r} : \widetilde{M} \to B$ has connected fibres and simply connected space $\widetilde{M}$. Therefore, due to the application of the exact homotopic sequence for this fibration we obtain that the base manifold $B$ is also simply connected.

For an arbitrary point $b \in B$ take $y \in \widetilde{r}^{-1}(b)$ and $x = \widetilde{\kappa}(y)$. Without loss generality we assume that there is a neighbourhood $U_i$, $x \in U_i$, from the $N$-cocycle $\{U_i, f_i, \{\gamma_{ij}\}\}_{i,j\in J}$ which defines $(M,F)$ and a neighbourhood $\widetilde{U}_i$, $y \in \widetilde{U}_i$, such that $\widetilde{\kappa}|_{\widetilde{U}_i} : \widetilde{U}_i \to U_i$ is a diffeomorphism.

Let $\widetilde{V}_i := \widetilde{r}(\widetilde{U}_i)$. Then there exists a diffeomorphism $\phi : \widetilde{V}_i \to V_i$ satisfying the equality $\phi \circ \widetilde{r}|_{\widetilde{U}_i} = f_i \circ \widetilde{\kappa}|_{\widetilde{U}_i}$. The diffeomorphism $\phi$ induces the Cartan geometry $\eta_{\widetilde{V}_i}$ on $\widetilde{V}_i$ such that $\phi$ becomes the isomorphism $(\widetilde{V}_i, \eta_{\widetilde{V}_i})$ and $(V_i, \xi_{V_i})$ in the category $\mathfrak{Car}$ of Cartan geometries. The direct check shows that by this way we define the Cartan geometry $\eta$ on $B$, and $\eta|_{\widetilde{V}_i} = \eta_{\widetilde{V}_i}$, $i \in J$.

Let us fix points $x_0 \in M$ and $y_0 \in \widetilde{\kappa}^{-1}(x_0) \in \widetilde{M}$. Then the fundamental group $\pi_1(M,x_0)$ acts on the universal covering space $\widetilde{M}$ as a deck transformation group $\widetilde{G} \cong \pi_1(M,x_0)$ of $\widetilde{\kappa}$. Since $\widetilde{G}$ preserves the inducted foliation $(\widetilde{M},\widetilde{F})$ formed by fibres of the fibration $\widetilde{r} : \widetilde{M} \to B$, then every $\widetilde{\psi} \in \widetilde{G}$ defines $\psi \in Diff(B)$ satisfying the relation $\widetilde{r} \circ \widetilde{\psi} = \psi \circ \widetilde{r}$. The map $\chi : \widetilde{G} \to \Psi : \widetilde{\psi} \to \psi$ is the group epimorphism. Observe that $\widetilde{G}$ is a subgroup of the automorphism group $A(\widetilde{M},\widetilde{F})$ of $(\widetilde{M},\widetilde{F})$ in the category $\mathfrak{CF}$. Therefore $\Psi$ is a subgroup of the automorphism group $Aut(B,\eta)$ in the category of Cartan geometries $\mathfrak{Car}$. The kernel $ker(\chi)$ of $\chi$ determines the quotient manifold $\widehat{M} := \widetilde{M}/ker(\chi)$ with the quotient map $\widehat{\kappa} : \widetilde{M} \to \widehat{M}$ and the quotient group $\widehat{G} := \widetilde{G}/ker(\chi)$ such that $M \cong \widehat{M}/\widehat{G}$. The quotient map $\kappa : \widehat{M} \to M$ is the required regular covering map, with $\widehat{G}$ acts on $\widehat{M}$ as a deck transformation group. The map $\widehat{G} \to \Psi : \widetilde{\psi}ker(\chi) \mapsto \chi(\widetilde{\psi})$, $\widetilde{\psi} \in \widetilde{G}$, is a group isomorphism.

Remark that the induced foliation $(\widehat{M},\widehat{F})$, $\widehat{F} = \kappa^* F$, is covered by a foliation $r : \widehat{M} \to B$ such that $\widetilde{r} = r \circ \widehat{\kappa}$.

Now the assertion (3) of Theorem 6 is easy proved with the application of Lemma 34.

## 5.3 Suspended foliations

Suspension foliation was introduced by A. Haefliger. Let $Q$ and $T$ be smooth connected manifolds. Denote by $\rho : \pi_1(Q,x) \to Diff(T)$ a group homomorphism. Let $G := \pi_1(Q,x)$ and $\Phi := \rho(G)$. Consider a universal covering map $\widetilde{p} : \widetilde{Q} \to Q$. A right action of the group $G$ on product of manifolds $\widetilde{Q} \times T$ is defined as follows:

$$\Theta : \widetilde{Q} \times T \times G \to \widetilde{Q} \times T : (x, t, g) \to (x \cdot g, \rho(g^{-1})(t)),$$

where the covering transformation $\widetilde{Q} \to \widetilde{Q} : x \to x \cdot g$ is induced by an element $g \in G$.

The quotient manifold $M := (\widetilde{Q} \times T)/G$ with the canonical projection

$$f_0 : \widetilde{Q} \times T \to M = (\widetilde{Q} \times T)/G$$

are determined.

Let $\Theta_g := \Theta|_{\widetilde{Q} \times \{t\} \times \{g\}}$. Since $\Theta_g(\widetilde{Q} \times \{t\}) = \widetilde{Q} \times \rho(g^{-1})(t)$ $\forall t \in T$, then the action of the discrete group $G$ on $(\widetilde{Q} \times T)$ preserves the trivial foliation $F := \{\widetilde{Q} \times \{t\} \mid t \in T\}$ of the product $\widetilde{Q} \times T$. Thus the projection $f_0 : \widetilde{Q} \times T \to M$ induced on the $M$ of the smooth foliation $F$. The pair $(M, F)$ is called *a suspended foliation* and is denoted by $Sus(T, Q, \rho)$. We accentuate that $(M, F)$ is covered by the trivial fibration $\widetilde{Q} \times T \to T$.

## 5.4 Proof of Theorem 8

Let $\eta$ be an effective Cartan geometry on a simply connected manifold $B$ and $\Psi$ be any countable subgroup of the automorphism Lie group $Aut(B, \eta)$. Therefore $\Psi$ has not more then countable generations $\psi_i$, $i \in \mathbb{N}$. Denote by $\mathbb{R}^2$ the usual plane and $C := \{(n, 0) \in \mathbb{R}^2 \mid n \in \mathbb{N}\}$. Consider the set $Q := \mathbb{R}^2 \setminus C$. Then the fundamental group $\pi_1(Q)$ of $Q$ is the free group with countable set of generators $\{\alpha_i \mid i \in \mathbb{N}\}$. Define a group homomorphism $\rho : \pi_1(Q) \to Aut(B, \eta)$ by the following equalities on generations $\rho(\alpha_i) = \psi_i$ for every $i \in \mathbb{N}$.

Then we construct the suspended foliation $(M, F) := Sus(B, Q, \rho)$ which is the 2-dimensional $(Aut(B, \eta), B)$-foliation covered by the trivial fibration $\mathbb{R}^2 \times B \to B$. Because $Im(\rho) = \Psi$ the group $\Psi$ is the global holonomy group of the Cartan foliation $(M, F)$.

## 5.5 Proof of Theorem 9

Let $(M, F)$ be a complete Cartan foliation of type $(G, H)$ whose transverse curvature is equal to zero. Then $(M, F)$ is modelled on the Cartan geometry $\xi_0 = (G(G/H, H), \beta_0)$ where $\beta_0$ is the Maurer–Cartan $\mathfrak{g}$-valued 1-form on the Lie group $G$. Hence $(M, F)$ is $(Aut(\xi_0), G/H)$-foliation. According to ([16], Proposition 3) the completeness of $(M, F)$ implies the existence of an Ehresmann connection for this foliation. As it is well known ([18], Theorem 2), any $(G, N)$-foliation with an Ehresmann connection is covered by a fibration. Therefore, $(M, F)$ is the Cartan foliation covered by a fibration and all statements of Theorem 6 are valid for it.

# 6 The structure of basic automorphism groups of foliations covered by fibrations

## 6.1 Properties of regular covering maps

**Definition 35.** Let $f : M \to B$ be a submersion. It is said that $\widehat{h} \in Diff(M)$ lying over $h \in Diff(B)$ relatively $f$ if $h \circ f = f \circ \widehat{h}$.

Let $\widetilde{\kappa} : \widetilde{K} \to K$ be the universal covering map, where $K$ and $\widetilde{K}$ are smooth manifolds. By analogy with Theorem 28.7 in [5], it is easy to show that for any $h \in$

$Diff(K)$ there exists $\widetilde{h} \in Diff(\widetilde{K})$ lying over $h$. It is well known that this fact is not true for regular covering maps in general. Proposition 36 solves the problem of lifting of transformations relatively arbitrary regular covering maps.

**Proposition 36.** *Let $\kappa : \widehat{K} \to K$ be a smooth regular covering map with the deck transformation group $\Gamma$ and $\widetilde{\kappa} : \widetilde{K} \to K$ be the universal covering map with the deck transformation group $\widetilde{\Gamma}$. Then*

*(1) A diffeomorphism $\widehat{h} \in Diff(\widehat{K})$ lies over some diffeomorphism $h \in Diff(K)$ if and only if it satisfies the equality $\widehat{h} \circ \Gamma = \Gamma \circ \widehat{h}$.*

*(2) For $h \in Diff(K)$ there exists $\widehat{h} \in Diff(\widehat{K})$ lying over $h$ if and only if there is $\widetilde{h}$ lying over $h$ relatively $\widetilde{\kappa} : \widetilde{K} \to K$ such that $\widetilde{h} \circ \widehat{\Gamma} = \widehat{\Gamma} \circ \widetilde{h}$ and $\widetilde{h} \circ \widetilde{\Gamma} = \widetilde{\Gamma} \circ \widetilde{h}$, where $\widehat{\Gamma}$ is the deck transformation group of the universal covering map $\widehat{\kappa} : \widetilde{K} \to \widehat{K}$, with $\widehat{\Gamma}$ is a normal subgroup of $\widetilde{\Gamma}$*

*(3) The set of all diffeomorphisms lying over $id_K$ relatively $\kappa : \widehat{K} \to K$ is coincided with the deck transformation group of $\Gamma \cong \widetilde{\Gamma}/\widehat{\Gamma}$.*

*(4) The subset of $h \in Diff(K)$ for which there exists a diffeomorphism $\widehat{h}$ of $\widehat{K}$ lies over $h$ forms a group which is isomorphic to the quotient group $N(\Gamma)/\Gamma$.*

*(5) Let $G$ be a group of diffeomorphisms of the manifold $K$ such that for every $g \in G$ there exists $\widehat{g} \in Diff(\widehat{K})$ lying over $g$ relatively $k : \widehat{K} \to K$. Then the full group $\widehat{G}$ of $\widehat{g} \in Diff(\widehat{K})$ lying over transformations from $G$ is isomorphic to the quotient group $\widehat{G}/\Gamma$.*

*Доказательство.* (1) Let $\widehat{h} \in Diff(\widehat{K})$ lies over $h$ relatively $\kappa : \widehat{K} \to K$, i.e. $h \circ \kappa = \kappa \circ \widehat{h}$. Then there exists $\widehat{h}^{-1} \in Diff(\widehat{K})$ and $h^{-1} \circ \kappa = \kappa \circ \widehat{h}^{-1}$, i.e., $\widehat{h}^{-1}$ lies over $h^{-1}$ relatively $\kappa$.

Let $\gamma$ be any element from $\Gamma$, then, according to the assumption (2), $\kappa \circ \gamma = \kappa$. Using this we get the following chain of equalities $\kappa \circ (\widehat{h} \circ \gamma \circ \widehat{h}^{-1}) = (\kappa \circ \widehat{h}) \circ \gamma \circ \widehat{h}^{-1} = (h \circ \kappa) \circ \gamma \circ \widehat{h}^{-1} = h \circ (\kappa \circ \gamma) \circ \widehat{h}^{-1} = h \circ \kappa \circ \widehat{h}^{-1} = (h \circ \kappa) \circ \widehat{h}^{-1} = (\kappa \circ \widehat{h}) \circ \widehat{h}^{-1} = \kappa$. Due to (2) this implies $\widehat{h} \circ \gamma \circ \widehat{h}^{-1} = \gamma' \in \Gamma$ and $\widehat{h} \circ \gamma = \gamma' \circ \widehat{h}$. Therefore $\widehat{h} \circ \Gamma \subset \Gamma \circ \widehat{h}$. Analogously, $\Gamma \circ \widehat{h} \subset \widehat{h} \circ \Gamma$. Thus, $\Gamma \circ \widehat{h} = \widehat{h} \circ \Gamma$.

(2) As $\kappa : \widehat{K} \to K$ is a regular covering map, $\widehat{\Gamma}$ is a normal subgroup of $\widetilde{\Gamma}$. Suppose that for $g \in Diff(K)$ there exists $\widehat{g} \in Diff(\widehat{K})$ lying over $g$. Consider the universal covering map $\widehat{\kappa} : \widetilde{K} \to \widehat{K}$. It is well known that there is the universal covering map $\widetilde{\kappa} : \widetilde{K} \to K$ satisfying the equality $\kappa \circ \widehat{\kappa} = \widetilde{\kappa}$. Hence there exists $\widetilde{g} \in Diff(\widetilde{K})$ over $\widehat{g}$ relatively $\widehat{\kappa}$. Note that $\widetilde{g}$ also lies over $g$ relatively $\widetilde{\kappa}$. Therefore, according to the proved above statement (1), $\widetilde{g}$ satisfies both equalities $\widetilde{g} \circ \widehat{\Gamma} = \widehat{\Gamma} \circ \widetilde{g}$ and $\widetilde{g} \circ \widetilde{\Gamma} = \widetilde{\Gamma} \circ \widetilde{g}$.

Converse. Suppose that for $h \in Diff(K)$ there exists $\widetilde{h} \in Diff(\widetilde{K})$ satisfying the equalities: $\widehat{k} \circ \widetilde{h} = \widehat{h} \circ \widehat{k}$, $\widetilde{h} \circ \widetilde{\Gamma} = \widetilde{\Gamma} \circ \widetilde{h}$ and $\widetilde{h} \circ \widehat{\Gamma} = \widehat{\Gamma} \circ \widetilde{h}$. Then, according to the assertion (1), there is $\widehat{h} \in Diff(\widehat{K})$ such that $\widehat{k} \circ \widetilde{h} = \widehat{h} \circ \widehat{k}$. Therefore, applying the equality $\kappa \circ \widehat{\kappa} = \widetilde{\kappa}$, for each $\widehat{x} \in \widehat{K}$, we get the chain of equalities

$$(\kappa \circ \widehat{h})(\widehat{x}) = \kappa \circ (\widehat{\kappa} \circ \widetilde{h})(\widehat{\kappa}^{-1}(\widehat{x})) = ((\kappa \circ \widehat{\kappa}) \circ \widetilde{h})(\widehat{\kappa}^{-1}(\widehat{x})) = (\widetilde{\kappa} \circ \widetilde{h})(\widehat{\kappa}^{-1}(\widehat{x})) =$$

$$= (h \circ \widetilde{\kappa})(\widehat{\kappa}^{-1}(\widehat{x})) = (h \circ (\kappa \circ \widehat{\kappa}))(\widehat{\kappa}^{-1}(\widehat{x})) = (h \circ \kappa)(\widehat{\kappa}(\widehat{\kappa}^{-1}(\widehat{x}))) = (h \circ \kappa)(\widehat{x}).$$

Hence, $\kappa \circ \widehat{h} = h \circ \kappa$, i.e. $\widehat{h}$ lies over $h$ relatively $\kappa$.

The statement (3) is obvious.

The assertion (4) is a corollary of (1) and (3).

(5) Let $G$ be the group of projections of $\widehat{G}$ and $f : \widehat{G} \to G : \widehat{h} \mapsto h$, where $h$ is the projection of $\widehat{h}$, is a group epimorphism, since $f$ is surjective by the condition. In according with the previous statement $Ker(f) = \Gamma$ and $G \cong \widehat{G}/\Gamma$. $\square$

## 6.2 Proof of Theorem 13

Suppose that a Cartan foliation $(M, F)$ is covered by fibration. By definition 5 the induced foliation $(\widetilde{M}, \widetilde{F})$ on the space of the universal covering $\widetilde{\kappa} : \widetilde{M} \to M$ is defined by a locally trivial fibration $\widetilde{r} : \widetilde{M} \to B$. Due to Theorem 6 the regular covering map $\kappa : \widehat{M} \to M$ and locally trivial fibration $r : \widehat{M} \to B$ are defined, where $B$ is a simply connected manifold with the inducted Cartan geometry $\eta$. Let $\Psi$ be the global holonomy group of $(M, F)$, then and $\Psi$ is isomorphic to the deck transformations group $G$ of $\kappa : \widehat{M} \to M$. Since the manifold $\widetilde{M}$ is simply connected, then there exists the universal covering map $\widehat{\kappa} : \widetilde{M} \to \widehat{M}$ satisfying the equality $\kappa \circ \widehat{\kappa} = \widetilde{\kappa}$. Let $\widetilde{G}$, $G$ and $\widehat{G}$ be the deck transformation groups of the covering maps $\widetilde{\kappa}$, $\kappa$ and $\widehat{\kappa}$ relatively, with $\Psi \cong G \cong \widetilde{G}/\widehat{G}$.

Let us consider the following preimages of the $H$-bundle $\mathcal{R}$ relatively $\widetilde{\kappa}$ and $\kappa$

$$\widetilde{\mathcal{R}} := \{(\widetilde{x}, u) \in \widetilde{M} \times \mathcal{R} \,|\, \widetilde{\kappa}(\widetilde{x}) = \pi(u)\} = \widetilde{\kappa}^* \mathcal{R} \;\; and$$

$$\widehat{\mathcal{R}} := \{(\widehat{x}, u) \in \widehat{M} \times \mathcal{R} \,|\, \kappa(\widehat{x}) = \pi(u)\} = \kappa^* \mathcal{R}.$$

Remark that the maps

$$\widetilde{\theta} : \widetilde{\mathcal{R}} \to \mathcal{R} : (\widetilde{x}, u) \mapsto (\widetilde{\kappa}(\widetilde{x}), u) \quad \forall (\widetilde{x}, u) \in \widetilde{\mathcal{R}},$$

$$\theta : \widehat{\mathcal{R}} \to \mathcal{R} : (\widehat{x}, u) \mapsto (\kappa(\widehat{x}), u) \quad \forall (\widehat{x}, u) \in \widehat{\mathcal{R}},$$

$$\widehat{\theta} : \widetilde{\mathcal{R}} \to \widehat{\mathcal{R}} : (\widetilde{x}, u) \mapsto (\widehat{\kappa}(\widetilde{x}), u) \quad \forall (\widetilde{x}, u) \in \widetilde{\mathcal{R}},$$

are regular covering maps with the deck transformation groups $\widetilde{\Gamma}$, $\Gamma$ and $\widehat{\Gamma}$, relatively, which are isomorphic to the relevant groups $\widetilde{G}$, $G$ and $\widehat{G}$, i.e. $\widetilde{\Gamma} \cong \widetilde{G}$, $\Gamma \cong G$ and $\widehat{\Gamma} \cong \widehat{G}$.

Let $(\widetilde{\mathcal{R}}, \widetilde{\mathcal{F}})$ and $(\widehat{\mathcal{R}}, \widehat{\mathcal{F}})$ be the corresponding lifted foliations. Since $(\widetilde{M}, \widetilde{F})$ and $(\widehat{M}, \widehat{F})$ are simple foliations, then $(\widetilde{\mathcal{R}}, \widetilde{\mathcal{F}})$ and $(\widehat{\mathcal{R}}, \widehat{\mathcal{F}})$ are also simple foliations, which are formed by locally trivial fibrations $\widetilde{\pi}_b : \widetilde{\mathcal{R}} \to \widetilde{W}$ and $\widehat{\pi}_b : \widehat{\mathcal{R}} \to \widehat{W}$. Hence $\mathfrak{g}_0\,(\widetilde{\mathcal{R}}, \widetilde{\mathcal{F}}) = 0$, $\mathfrak{g}_0(\widehat{\mathcal{R}}, \widehat{\mathcal{F}}) = 0$, and $\widetilde{W} = \widetilde{\mathcal{R}}/\widetilde{\mathcal{F}}$, $\widehat{W} = \widehat{\mathcal{R}}/\widehat{\mathcal{F}}$ are manifolds.

Since the fibrations $\widetilde{r} : \widetilde{M} \to B$ and $r : \widehat{M} \to B$ have the same base $B$, each leaf of the foliation $(\widetilde{M}, \widetilde{F})$ is invariant relatively the group $\widehat{G}$, i.e. $\widehat{G} \subset A_L(\widetilde{M}, \widetilde{F})$. Therefore $\widehat{\Gamma} \subset A_L(\widetilde{\mathcal{R}}, \widetilde{\mathcal{F}})$ and the leaf spaces $\widetilde{\mathcal{R}}/\widetilde{\mathcal{F}} = \widetilde{W}$ and $\widehat{\mathcal{R}}/\widehat{\mathcal{F}} = \widehat{W}$ are coincided, i.e. $\widetilde{W} = \widehat{W}$. Consequently basic automorphism groups $A_B(\widetilde{\mathcal{R}}, \widetilde{\mathcal{F}})$ and $A_B(\widehat{\mathcal{R}}, \widehat{\mathcal{F}})$ may be identified. Further we put $A_B(\widetilde{\mathcal{R}}, \widetilde{\mathcal{F}}) = A_B(\widehat{\mathcal{R}}, \widehat{\mathcal{F}})$.

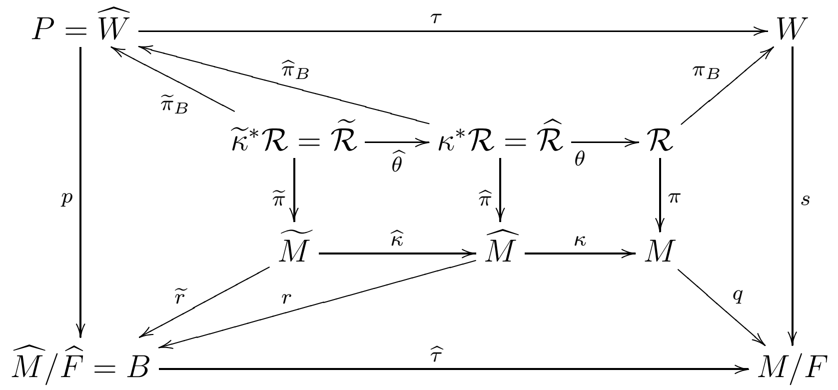
According to the conditions of Theorem 13, $\Psi$ is a discrete subgroup of the Lie group $Aut(B, \eta)$. Let $N(\Psi)$ be the normalizer of $\Psi$ in the Lie group $Aut(B, \eta) \cong A^H(W, \widetilde{\beta})$. Hence, $N(\Psi)$ is a closed Lie subgroup of the Lie group $Aut(B, \eta)$ and the quotient group $N(\Psi)/\Psi$ is also a Lie group.

Let $\pi : \mathcal{R} \to M$ be the projection of the foliated bundle over $(M, F)$. Due to Theorem 11 the discreteness of the global holonomy group $\Psi$ implies that the structural Lie algebra $\mathfrak{g}_0$ of the Cartan foliations $(M, F)$ is zero. Therefore the lifted foliation $(\mathcal{R}, \mathcal{F})$ is formed by fibres of the basic fibration $\pi_b : \mathcal{R} \to W$.

Observe that there exists a map $\tau : \widehat{W} \to W$ satisfying the equality $\tau \circ \widehat{\pi}_b = \theta \circ \pi_b$. It is easy to show that $\tau : \widehat{W} \to W$ is a regular covering map with the deck transformations group $\Phi$, $\Phi \subset A^H(\widehat{W}, \widehat{\beta})$, which is naturally isomorphic to the groups $\Psi$, $G$ and $\Gamma$.

Denote by $\eta = (P(B,H), \omega)$ the Cartan geometry with the projection $p : P \to B$ onto $B$ determined in the proof of Theorem 6. Remark that $\widehat{W} = P$ is the space of the $H$-bundle of the Cartan geometry $\eta$.

Since $\kappa : \widehat{M} \to M$ and $\pi : \mathcal{R} \to M$ are morphisms of the following foliations $\kappa : (\widehat{M}, \widehat{F}) \to (M, F)$ and $\theta : (\widehat{\mathcal{R}}, \widehat{\mathcal{F}}) \to (\mathcal{R}, \mathcal{F})$ in the category of the foliations $\mathfrak{Fol}$, then maps $\widehat{\tau} : B \to M/F$ and $s : W \to W/H \cong M/F$ are defined, and the following diagram



is commutative.

Due to Proposition 31 there are the Lie group isomorphisms

$$\varepsilon : A_B(M, F) \to Im(\varepsilon) \subset A^H(W, \widetilde{\beta}) \; and$$

$$\widehat{\varepsilon} : A_B(\widetilde{M}, \widetilde{F}) = A_B(\widehat{M}, \widehat{F}) \to Im(\widehat{\varepsilon}) \subset A^H(\widehat{W}, \widehat{\beta}).$$

Let us define a map $\Theta : Im(\varepsilon) \to N(\Phi)/\Phi$ by the following a way. Take any $h \in Im(\varepsilon) \subset A^H(W, \widetilde{\beta})$. Denote the element $\varepsilon^{-1}(h) \in A_B(M, F)$ by $f \cdot A_L(M, F) \in A_B(M, F)$, where $f \in A(M, F)$. Since $\widetilde{\kappa} : \widetilde{M} \to M$ is the universal covering map there exists $\widetilde{f} \in Diff(\widetilde{M})$ lying over $f$ relatively $\widetilde{\kappa}$. It not difficult to see tha $\widetilde{f} \in A(\widetilde{M}, \widetilde{F})$. Hence $\widetilde{f} \circ A_L(\widetilde{M}, \widetilde{F}) \in A_B(\widetilde{M}, \widetilde{F})$. Consider $\widehat{h} := \widehat{\varepsilon}(\widetilde{f} \cdot A_L(\widetilde{M}, \widetilde{F})) \in Im(\widehat{\varepsilon}) \subset A^H(\widehat{W}, \widehat{\beta})$. The direct check shows that $\widehat{h}$ lies over $h$ relatively $\tau$. Remind that $\Phi$ is the deck transformation group of the covering map $\tau : \widehat{W} \to W$. Applying the statement (1) of Proposition 36 we get that $\widehat{h} \in N(\Phi)$ and the set of all automorphisms in $Im(\widehat{\varepsilon})$ lying over $h$ is equal to the set of transformations from the class $\widehat{h} \cdot \Phi$. Let us put $\Theta(h) := \widehat{h} \cdot \Phi \in N(\Phi)/\Phi$. It is easy to check that the map $\Theta : Im(\varepsilon) \to N(\Phi)/\Phi$ is a group monomorphism.

The effectiveness of the Cartan geometry $\eta = (P(B,H), \omega)$ on $B$, where $P = \widehat{W}$, implies the existence of the Lie group isomorphism $\sigma : A^H(\widehat{W}, \widehat{\beta}) \to Aut(B, \eta)$ (see Remark 16). Observe that $\sigma(\Phi) = \Psi$ and $\sigma(N(\widetilde{\Phi})) = N(\Psi)$, hence there exists the inducted Lie group isomorphism $\widetilde{\sigma} : N(\Phi)/\Phi \to N(\Psi)/\Psi$. Thus, the composition of the Lie group monomorphisms

$$\delta := \widetilde{\sigma} \circ \Theta \circ \varepsilon : A_B(M, F) \to N(\Psi)/\Psi$$

is the required Lie group monomorphism. Due to uniqueness of the Lie group structure in $A_B(M, F)$, in conforming with the proof of Theorem 1, the image $Im(\delta)$ is an open-closed subgroup of the Lie group $N(\Psi)/\Psi$.

## 6.3 Proof of Theorem 14

1. In accordance with the condition of Theorem 14, $(M, F)$ is an $\mathfrak{M}$-complete Cartan foliation, and the distribution $\mathfrak{M}$ is integrable. In this case, there is $q$-dimensional

foliation $(M, F^t)$ such that $TF^t = \mathfrak{M}$. Let $\widetilde{\kappa} : \widetilde{M} \to M$ be the universal covering map. As is known ([16], Proposition 2), $\mathfrak{M}$ is an integrable Ehresmann connection for the foliation $(M, F)$.

In this case, according to the decomposition theorem belonging to S. Kashiwabara [8], the universal covering manifold has the form $\widetilde{M} = \widetilde{Q} \times B$, and the lifted foliations are $\widetilde{F} = \widetilde{\kappa}^* F = \{\widetilde{Q} \times \{y\} \,|\, y \in B\}$, $\widetilde{F}^t = \widetilde{\kappa}^* F^t = \{\{z\} \times B \,|\, z \in \widetilde{Q}\}$. Hence $(M, F)$ is covered by fibration $\widetilde{r} : \widetilde{Q} \times B \to B$, where $B$ is a simply connected manifold. In this case, by the same way as in the proof of Theorem 6, the Cartan geometry $\eta$ is induced on $B$ such that $(M, F)$ becomes $(Aut(B, \eta), B)$-foliation.

2. Let $\Psi$ be the global holonomy group of this foliation. Suppose now that the normalizer $N(\Psi)$ is equal to the centralizer $Z(\Psi)$ of $\Psi$ in the group $Aut(B, \eta)$.

Let us fix points $x_0 \in M$ and $(z_0, y_0) \in \widetilde{\kappa}^{-1}(x_0) \in \widetilde{M}$. Then the fundamental group $\pi_1(M, x_0)$ acts on the universal covering space $\widetilde{M} = \widetilde{Q} \times B$ as the deck transformation group $\widetilde{G} \cong \pi_1(M, x_0)$ of $\widetilde{\kappa}$. Since $\widetilde{G}$ preserves both the inducted foliations $(\widetilde{M}, \widetilde{F})$ and $(\widetilde{M}, \widetilde{F}^t)$, then every $\widetilde{g} \in \widetilde{G}$ may be written in the form $\widetilde{g} = (\psi^t, \psi)$, where $\psi^t$ forms a subgroup $\Psi^t$ in $Diff(\widetilde{Q})$, $\psi \in \Psi$, and $\widetilde{g}(z, y) = (\psi^t(z), \psi(y))$, $(z, y) \in \widetilde{Q} \times B$. The maps $\widetilde{\chi} : \widetilde{G} \to \Psi : \widetilde{g} \to \psi$ and $\widetilde{\chi}^t : \widetilde{G} \to \Psi^t : \widetilde{g} \to \psi^t$ are the group epimorphisms.

Let $h$ be any element from $N(\Psi)/\Psi$. Since $N(\Psi) = Z(\Psi)$, we have the following chain of equalities

$$\widetilde{g} \circ (id_{\widetilde{Q}}, h) = (\psi^t, \psi) \circ (id_{\widetilde{Q}}, h) = (\psi^t \circ id_{\widetilde{Q}}, \psi \circ h) =$$

$$(id_{\widetilde{Q}} \circ \psi^t, h \circ \psi) = (id_{\widetilde{Q}}, h) \circ (\psi^t, \psi) = (id_{\widetilde{Q}}, h) \circ \widetilde{g}$$

for any $\widetilde{g} = (\psi^t, \psi) \in \widetilde{G}$, i.e. $\widetilde{G} \circ (id_{\widetilde{Q}}, h) = (id_{\widetilde{Q}}, h) \circ \widetilde{G}$. Therefore, by the statement (1) of Proposition 36 for the deck transformation group $\widetilde{G}$, there exists $\widetilde{h} \in Diff(M)$ such that $(id_{\widetilde{Q}}, h)$ lies over $\widetilde{h}$ relatively to $\widetilde{\kappa} : \widetilde{M} \to M$.

Using $(id_{\widehat{Q}}, h) \in A(\widetilde{M}, \widetilde{F})$ it is not difficult to check that $\widetilde{h} \in A(M, F)$. Hence, $\varepsilon(\widetilde{h} \cdot A_L(M, F)) = h$. This means that $\varepsilon : A_B(M, F) \to N(\Psi)/\Psi$ is surjective. Thus, $\varepsilon$ is the group isomorphism.

# 7 Examples of the calculation of the basic automorphisms groups

**Definition 37.** Let $\xi = (P(N, H), \omega)$ is arbitrary Cartan geometry of the type $(G, H)$, of the effectiveness of which is not assumed. The group

$$\mathrm{Gauge}(\xi) := \{\Gamma \in A(\xi) \mid p \circ \Gamma = p\}$$

is called *of the gauge transformation group of the Cartan geometry* $\xi$.

*Example* 1. Let $G$ be a Lie group and $H$ be a closed subgroup of $G$. Denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of Lie groups $G$ and $H$ relatively. Suppose that the kernel of the pair of Lie groups $(G, H)$ is equal to the intersection $K = Z(G) \cap Z(H)$ of the centers of the groups $G$ and $H$. Denote by $\omega_G$ the Maurer-Cartan $\mathfrak{g}$-valued 1-form on the Lie group $G$. Then $\xi^0 = (G(G/H, H), \omega_G)$ is the Cartan geometry, and its transverse curvature is zero. Consider any smooth manifold $L$ . Denote by $M$ the product of manifolds $M = L \times (G/H)$, and $F = \{L \times \{x\} \,|\, x \in G/H\}$. Then $(M, F)$ is the trivial transverse homogeneous foliation with the transverse Cartan geometry $\xi^0$. Because the foliation $(M,F)$ is trivial, the group $A_B(M,F)$ is coincided with the automorphisms group $Aut(\xi^0)$ of the Cartan geometry $\xi^0$ in the category $\mathfrak{Car}$.

Any left action $L_g$, $g\in G$, of the Lie group $G$ satisfies the conditions: $L_g^*\omega_G=\omega_G$ and $L_g\circ R_a=R_a\circ L_g$ $\forall a\in G$. Therefore, $L_g\in Aut(\xi^0)$ and $\dim(Aut(\xi^0))=\dim(G)=\dim(\mathfrak{g})$. By assumption, the kernel of the pair $(G,H)$ equal to $K=Z(G)\cap Z(H)$, hence $Gauge(\xi^0)=\{L_b\,|\,b\in\ K\}$. Thus, the basic automorphisms group $A_B(M,F)$ is equal to the quotient $Aut(\xi^0)/Gauge(\xi^0)\cong G/K$, and $\dim(A_B(M,F))=\dim(\mathfrak{g})-\dim(\mathfrak{k})$, where $\mathfrak{k}$ is the algebra Lie of the kernel $K$.

Example 1 shows that the estimation (1.1) of the dimension group $Aut_B(M,F)$ in Theorem 1 is exact.

*Example* 2. Let

$$\left\{G:=\begin{pmatrix}1&x&y\\0&1&x\\0&0&1\end{pmatrix}\,|\,x,y\in\mathbb{R}^1\right\},\quad K:=\left\{\begin{pmatrix}1&0&y\\0&1&0\\0&0&1\end{pmatrix}\,|\,y\in\mathbb{R}^1\right\}.$$

Then $G$ is an Abelian Lie group and $K$ is a connected closed subgroup of the group Lie $G$. Hence $K=Z(G)\cap\ Z(K)$. Let $\xi^0=(G(G/K,K),\omega_G)$ is canonical Cartan geometry with projection $p:G\to G/K$.

The foliation $(G,F)$, where $F=\{gK\,|\,g\in G\}$, is a proper Cartan foliation with transverse Cartan geometry $\xi^0$. Using Proposition 32 we see that $\mathcal{R}=G$ and $H=\{e\}$. Therefore the basic automorphisms group $A_B(M,F)$ is isomorphic to the Lie group $Aut(G/K,\xi^0)\cong G/K$. Thus $A_B(M,F)\cong G/K$.

Since $K=H$, then $\dim(A_B(M,F))=\dim(G)-\dim(K)=\dim\mathfrak{g}-\dim\mathfrak{k}$ and the estimation (1.1) of Theorem 1 is exact.

*Example* 3. Let $G$ be the similar group of the Euclidean space $\mathbb{E}^q$, $q\geq 1$. Then $G=CO(q)\ltimes\mathbb{R}^q$ is the semidirect product of the conformal group $CO(q)$ and the group $\mathbb{R}^q$. Let $H=CO(q)$ and $p\colon G\to G/H=\mathbb{E}^q$ be the canonical principal $H$-bundle. Let $\mathfrak{g}$ be the Lie algebra of the Lie group $G$, and $\omega_G$ be the Maurer-Cartan $\mathfrak{g}$-valued 1-form on $G$. Then $\xi=(G(\mathbb{E}^q,H),\omega_G)$ is an effective Cartan geometry. Foliations with this transverse geometry $(\mathbb{E}^q,\xi)$ are called *transversally similar foliations* [16].

Let $Q$ be a smooth $p$-dimensional manifold whose fundamental group $\pi_1(Q,x)$ contains an element $\alpha$ of infinite order. For an arbitrary natural number $q\geq 1$, denote by $\mathbb{E}^q$ the $q$-dimensional Euclidean space.

Define a group homomorphism $\rho\colon\pi_1(Q,x)\to Diff(\mathbb{E}^q)$ by setting $\rho(\alpha)=\psi$, where $\psi$ is the homothety transformation of the Euclidean space $\mathbb{E}^q$ with the coefficient $\lambda\neq 1$, i. e. $\psi(x)=\lambda x$ $\forall x\in\mathbb{E}^q$, and $\rho(\beta)=\mathrm{id}_{\mathbb{E}^q}$ for any element $\beta\in\pi_1(Q,x)$ such that $\beta\neq\alpha^k$ with every integer $k$. Then $(M,F)=\mathrm{Sus}(\mathbb{E}^q,Q,\rho)$ is a proper transversally similar foliation with a unique closed leaf diffeomorphic to the manifold $Q$.

Due to $N(\Psi)=Z(\Psi)$, according to Theorem 14, we get $A_B(M,F)\cong N(\Psi)/\Psi$. The foliation $(M,F)$ is covered by the fibration $\widetilde{Q}\times\mathbb{E}^q\to\mathbb{E}^q$ where $\widetilde{Q}\to Q$ is the universal covering map. Hence $\Psi:=\rho(\pi_1(Q,x))\cong\mathbb{Z}$ is the global holonomy group of $(M,F)$ and $K=H$ is the kernel of the pair $(G,H)$. Thus the assumption of which was made in Example 1 is realized.

In our case $\Psi=\langle\psi\rangle$ and $N(\Psi)=CO(q)=\mathbb{R}^+\cdot O(q)$, therefore $A_B(M,F)\cong U(1)\times O(q)$, where $U(1)\cong\mathbb{R}^+/\Psi$ is the compact 1-dimensional Abelian group.

If $q=1$, then $O(q)=\mathbb{Z}_2$ and $A_B(M,F)\cong U(1)\times\mathbb{Z}_2$.

Thus $\dim(A_B(M,F))=\dim\mathfrak{h}$ and the estimate (1.2) in Theorem 1 is exact.

*Example* 4. Let $\mathbb{E}^2=(\mathbb{R}^2,g)$ be an Euclidean plane with an Euclidean metric $g$. Let $\psi$ be the rotation of the Euclidean plane $\mathbb{E}^2$ around the point $0\in\mathbb{E}^2$ by the angle $\delta=2\pi r$. Denote by $\mathfrak{I}(\mathbb{E}^2)$ the full isometry group of $\mathbb{E}^2$. It is well known that $\mathfrak{I}(\mathbb{E}^2)\cong O(2)\ltimes\mathbb{R}^2$.

Let $\rho\colon \pi_1(S^1, b) \cong \mathbb{Z} \to \mathfrak{I}(\mathbb{E}^2)$ be defined by the equality $\rho(1) := \psi$, $1 \in \mathbb{Z}$. Then we have the suspended Riemannian foliation $(M, F) := Sus(\mathbb{E}^2, S^1, \rho)$. This foliation has a unique closed leaf which is compact.

Due to $N(\Psi) = Z(\Psi) = O(2)$ Theorem 14 is applicable. Consequently, $A_B(M, F) \cong N(\Psi)/\Psi = O(2)/\Psi$. Hence $A_B(M, F)$ admits a Lie group structure if and only if $\Psi$ is a closed subgroup of $O(2)$ or, equivalent, when $\delta = 2\pi r$ for some rational number $r$.

If $\delta = 2\pi r$, where $r$ is a nonzero rational number, then $A_B(M, F) \cong O(2)$.

*Example* 5. Consider the standard 2-dimension torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ and call the pair of vectors $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ by the standard basis of the tangent vector space $T_x\mathbb{T}^2$ with $x \in \mathbb{T}^2$. Let $\Omega : \mathbb{R}^2 \to \mathbb{T}^2$ be the quotient map, which is the universal covering of the torus. Denote by $f_A$ the Anosov automorphism of the torus $\mathbb{T}^2$ determined by the matrix $A \in SL(2, \mathbb{Z})$, while by $E$ the identity $2 \times 2$ matrix.

Let $g$ be the flat Lorentzian metric on the torus $\mathbb{T}^2$ given in the standard basis by the matrix $\lambda \begin{pmatrix} 2 & m \\ m & 2 \end{pmatrix}$, where $\lambda$ is any non zero real number and $m \in \mathbb{Z}$, $|m| > 2$. Introduce notations $\mathfrak{I}(\mathbb{T}^2, g)$ for the full isometry group of this Lorentzian torus $(\mathbb{T}^2, g)$ and $\mathfrak{I}_0(\mathbb{T}^2, g)$ for the stationary subgroup of the group $\mathfrak{I}(\mathbb{T}^2, g)$ at point $0 = \Omega(0)$, $0 = (0, 0) \in \mathbb{R}^2$. As is known ([19], Example 3), $\mathfrak{I}(\mathbb{T}^2, g) = \mathfrak{I}_0(\mathbb{T}^2, g) \ltimes \mathbb{Z} \times \mathbb{Z}$, where the group $\Phi_0 := \mathfrak{I}_0(\mathbb{T}^2, g)$ is generated by $f_A$, $f_{\widetilde{A}}$ and $-E$, $A = \begin{pmatrix} m & 1 \\ -1 & 0 \end{pmatrix}$ and $\widetilde{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, hence $\mathfrak{I}(\mathbb{T}^2, g) \cong (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}) \ltimes T^2$.

Let $Q = S^1$ and $T = \mathbb{T}^2$. Define the group homomorphism $\rho : \pi_1(S^1) \cong \mathbb{Z} \to \mathfrak{I}(\mathbb{T}^2, g)$ by the equality $\rho(k) := (f_A)^k$, $k \in \mathbb{Z}$. Then the suspended foliation $(M, F) := Sus(\mathbb{T}^2, S^1, \rho)$ is Lorentzian, and its global holonomy group $\Psi$ is the group of all transformations lying over the group $\Phi :=< f_A >$ relatively the universal covering map $\Omega : \mathbb{R}^2 \to \mathbb{T}^2$.

Elements of affine group $Aff(A^2)$ will be denoted by $< C, c >$, $C \in GL(2, \mathbb{R})$, $c \in R^2$, in compliance with $Aff(A^2) = GL(2, \mathbb{R}) \ltimes R^2$. The composition of transformations $< C, c >$ and $< D, d >$ from $Aff(A^2)$ has the following form $< C, c >< D, d >=< CD, Cd + c >$.

The check using the Proposition 36 shows that $\Psi = \Psi_0^0 \ltimes (\mathbb{Z} \times \mathbb{Z})$, where the group $\Psi_0^0$ is generated by matrix $A$, i.e. $\Psi_0^0 \cong \Phi$. Let $\Gamma := \mathbb{Z} \times \mathbb{Z} \subset \mathfrak{I}(\mathbb{E}^2)$.

Consider any $< C, c >\in N(\Psi)$, then for every $< E, a >\in \Gamma$ there are $< D, d >, < K, b >\in \Psi$ such that

$$< C, c >< E, a >=< D, d >< C, c >, \tag{7.1}$$

$$< C, c >< K, d >=< E, a >< C, c > . \tag{7.2}$$

Hence $D = E$, $K = E$ and $< C, c >\in N(\Gamma)$. Consequently $N(\Psi) \subset N(\Gamma)$ and, due to $\Gamma$ is the deck transformation group of $\Omega$, by the statement (2) of Proposition 36, the following map

$$\alpha : N(\Psi) \to \mathfrak{I}(\mathbb{T}^2, g) : \widehat{h} \mapsto h,$$

where $\widehat{h}$ lies over $h$ relatively $\Omega : \mathbb{R}^2 \to \mathbb{T}^2$, is defined and it is a group homomorphism.

The relations (7.1) and (7.2) imply also that $< C, 0 >\in N(\Gamma)$. Therefore $f_C \in \Phi_0 := \mathfrak{I}_0(\mathbb{T}^2, g)$ and $C \in \Psi_0$, where $\Psi_0$ is the subgroup of $N(\Psi)$ generated by matrixes $A$, $\widetilde{A}$ and $-E$, i.e. $\Psi_0 \cong \Phi_0$. Thus, the stationary subgroup $N(\Psi)_0$ at $0 \in \mathbb{R}^2$ of the normalizer $N(\Psi)$ is equal to $\Psi_0$.

Since $\alpha(\Psi) = \Phi$, then the group homomorphism $\alpha$ has the property $\alpha(N(\Psi)) = N(\Phi)$.

Let us compute the normalizer $N(\Phi)$ of $\Phi$ in the group $\mathfrak{I}(\mathbb{T}^2, g)$. Take any $< D, d >$ from $N(\Phi)$. Then there is $k \in \mathbb{Z} \setminus \{0\}$ such that

$$< D, d >< A, 0 >=< A^k, 0 >< D, d >,$$

consequently $A^k d = d$, i.e. 1 is the eigenvalue of $A^k$. Since $A^k$ is an Anosov automorphism, then its eigenvalues are irrational. Thus, it is necessary $d = 0$, hence $N(\Phi) \subset \Phi_0$. Observe that $A\widetilde{A} = \widetilde{A}A^{-1}$ and $A(-E) = (-E)A$, therefore, $N(\Phi) = \Phi_0$. Thus, $N(\Phi)/\Phi = \Phi_0/\Phi \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, i.e. $N(\Phi)/\Phi \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

By Theorem 13 there is the group monomorphism $\varepsilon : A_B(M, F) \to N(\Psi)/\Psi$ and the image $Im(\varepsilon)$ is an open-closed subgroup of $N(\Psi)/\Psi$. Show that $\varepsilon$ is a surjection.

Note that $N(\Psi) = \Psi_0 \ltimes \mathbb{Z} \times \mathbb{Z}$ and the quotient group $N(\Psi)/\Psi$ is generated by transformations $< \widetilde{A}, 0 >$ and $< -E, 0 >$ of $\mathbb{R}^2$. Every transformations of product $\mathbb{R}^1 \times \mathbb{R}^2$ conserving this product may be written as a pair $(g_1, g_2)$, where $g_1 \in Diff(\mathbb{R}^1)$, $g_2 \in Diff(\mathbb{R}^2)$, and $(g_1, g_2)(t, z) = (g_1(t), g_2(z))$, $(t, z) \in \mathbb{R}^1 \times \mathbb{R}^2$. Let $\widetilde{G} = \{(\widetilde{g}_1, \widetilde{g}_2)\}$ be the group of covering transformations of the universal covering map $\widetilde{\kappa} : \mathbb{R}^1 \times \mathbb{R}^2 \to M$, then $\widetilde{g}_1 : \mathbb{R}^1 \to \mathbb{R}^1 : t \mapsto t + m$, $m \in \mathbb{Z}$, $\widetilde{g}_2 \in \Psi$. Let $\widetilde{f} : \mathbb{R}^1 \to \mathbb{R}^1 : t \mapsto -t$ be the diffeomorphism of $\mathbb{R}^1$. Observe that $(\widetilde{f}, \widetilde{A}) \in A(\mathbb{R}^3, \widetilde{F}_{tr})$, where $\widetilde{F}_{tr} = \{\mathbb{R}^1 \times \{z\} \,|\, z \in \mathbb{R}^2\}$, and $(\widetilde{f}, \widetilde{A})$ lies over $(\widetilde{A}, 0)$ of $\mathbb{R}^2$ relatively the canonical projection onto second multiplier $\widetilde{r} : \mathbb{R}^1 \times \mathbb{R}^2 \to \mathbb{R}^2$. Since $(f, A) \circ (\widetilde{f}, \widetilde{A}) = (\widetilde{f}, \widetilde{A}) \circ (f^{-1}, A^{-1})$, we see that $(\widetilde{f}, \widetilde{A}) \circ \widetilde{G} = \widetilde{G} \circ (\widetilde{f}, \widetilde{A})$. Hence, there exists $\widetilde{h} \in A(M, F)$ such that $(\widetilde{f}, \widetilde{A})$ lies over $\widetilde{h}$ relative $\widetilde{\kappa}$. It means that $\varepsilon(\widetilde{h} \cdot A_L(M, F)) =< \widetilde{A}, 0 >$. The existence $h' \in A(M, F)$ such that $\varepsilon(h' \cdot A_L(M, F)) =< -E, 0 >$ is cheeked by the same way as in the proof of Theorem 14. Therefore the group homomorphism $\varepsilon : A_B(M, F) \to N(\Psi)/\Psi$ is surjection, hence $\varepsilon$ is the group isomorphism.

Due to the following chain of group isomorphisms $A_B(M, F) \cong N(\Psi)/\Psi$ $\cong (N(\Psi)/\Gamma)/(\Psi/\Gamma) \cong N(\Phi)/\Phi \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, we have

$$A_B(M, F) \cong \mathbb{Z}_2 \times \mathbb{Z}_2.$$

*Remark* 38. It is well known (see, for example, [13], Lemma 3.3) that the set of periodic orbits of a Anosov automorphism of the torus $\mathbb{T}^2$ is countable. Therefore the foliation $(M, F)$ constructed in Example 5 has a countable set of closed leaves and according to the item $(b)$ of Theorem 1 its basic automorphism group $A_B(M, F)$ is a discrete Lie group. Our result $A_B(M, F) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ illustrates this assertion.

# Список литературы


[1] A. V. Bagaev, N. I. Zhukova, The isometry group of Riemannian orbifolds, *Siberian Mathematical Journal* **48**:4 (2007), 579–592.

[2] I. V. Belko, Affine transformations of a transversal projectable connection on a foliated manifold, *Mathematics of the USSR-Sbornik* **45**:2 (1983), 191–203.

[3] R. Blumenthal, Cartan submersion and Cartan foliations, *Ill. J. Math.* **31**:2 (1987), 327–343.

[4] R. A. Blumenthal, J. J. Hebda, Ehresmann connections for foliations, *Indiana Univ. Math. J.* **33**:4 (1984), 597–611.

[5] H. Busemann, *The geometry of geodesics.* Academic Press, New York, 2011.

[6] A. Cap, J. Slovak, *Parabolic Geometries I: Background and General Theory,* Mathematical Surveys and Monographs **154**. AMS: Publishing House, 2009.

[7] L. Conlon, Transversally parallelizable foliations of codimension 2, *Trans. Amer. Math. Soc.* **194** (1974), 79–102.

[8] S. Kashiwabara, The decomposition of differential manifolds and its applications, *Tohoku Math. J.* **11** (1959), 43–53.

[9] S. Kobayashi, *Transformation group in differential geometry.* Springer-Verlag, New York, 1995.

[10] S. Kobayashi, K. Nomizu, *Foundations of differential geometry, I.* Intersience Publ., New York-London, 1969.

[11] J. Leslie, A remark on the group of automorphisms of a foliation having a dense leaf, *J. Differ. Geom.* **7** (1972), 597–601.

[12] P. Molino, *Riemannian foliations.* Progress in Math. Birkhauser, Boston, 1988.

[13] Z. Nitezki, *Differentiable Dynamics: an Introduction to the Orbit Structure of Diffeomorphisms.* The MIT Press: Cambridge, Mass., 1971.

[14] Sharpe R.W.*Differential Geometry: Cartan's Generalization of Klein's Erlangen Progpam.* Graduate Texts in Mathematics **166**. Springer-Verlag, New York, 1997.

[15] I. Tamura, *Topology of foliations.* Translations of math. monograph. AMS: Publishing House, New York, 1992.

[16] N. I. Zhukova, Minimal sets of Cartan foliations, *Proc. of the Steklov Inst. of Math.* **256** (2007), 105–135.

[17] N. I. Zhukova, Complete foliations with transverse rigid geometries and their basic automorphisms, *Bulletin of Peoples' Friendship University of Russia. Ser. Math. Information Sci. Phys.* **2** (2009), 14–35.

[18] N. I. Zhukova, Global attractors of complete conformal foliations, *Sbornik: Mathematics* **203**:3 (2012), 380–405.

[19] N. I. Zhukova, E. A. Rogozhina, Classification of compact Lorentzian 2-orbifolds with noncompact isometry group, *Siberian Mathematical Journal* **53**:6 (2012), 1037–1050.